An isomorphism theorem for models of Weak König's Lemma without primitive recursion

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Abstract

We prove that if $(M, \mathcal{X})$ and $(M, \mathcal{Y})$ are countable models of the theory WKL_0^* such that $\mathrm{I}\Sigma_1(A)$ fails for some $A \in \mathcal{X} \cap \mathcal{Y}$, then $(M, \mathcal{X})$ and $(M, \mathcal{Y})$ are isomorphic. As a consequence, the analytic hierarchy collapses to Δ^1_1 provably in $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1$, and WKL is the strongest Π^1_2 statement that is Π^1_1-conservative over $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$.

Applying our results to the Δ^0_n-definable sets in models of $\mathrm{RCA}_0^* + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ that also satisfy an appropriate relativization of Weak König's Lemma, we prove that for each $n \geqslant 1$, the set of Π^1_2 sentences that are Π^1_1-conservative over $\mathrm{RCA}_0^* + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ is c.e. In contrast, we prove that the set of Π^1_2 sentences that are Π^1_1-conservative over $\mathrm{RCA}_0^* + \mathrm{B}\Sigma^0_n$ is Π_2-complete. This answers a question of Towsner.

We also show that $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ is Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$ if and only if it is conservative over $\mathrm{B}\Sigma^0_2$ with respect to $\forall\Pi^0_5$ sentences.

Keywords models of arithmetic, quantifier elimination, automorphism, conservation theorems, reverse mathematics, fragments of arithmetic, Ramsey's theorem, negated induction

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In this paper, we investigate a new model-theoretic argument that can be used to study some fragments of second-order arithmetic. It is a cliché that classical model-theoretic techniques are of limited use in understanding models of arithmetic, and in particular it is well-known that commonly studied theories of arithmetic do not have nice model-theoretic properties such as model completeness and quantifier elimination. Nevertheless, we use an automorphism-based argument to obtain a kind of model completeness result for a specific fragment of second-order arithmetic with (partially) negated induction. This result can be applied to obtain new information both about general properties of some theories of second-order arithmetic, including true ones, and about specific principles considered in reverse mathematics.

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The main theorem of the paper concerns the axiomatic theory known as WKL_0^*. This is a fragment of second-order arithmetic consisting of the theory RCA_0^* – that is, Δ_1^0-comprehension, Δ_1^0-induction, and the totality of exponentiation – and the additional axiom WKL expressing a compactness principle called Weak König's Lemma, which says that any infinite 0–1 tree has an infinite path. Compared to the usual base theory considered in reverse mathematics, RCA_0, the system WKL_0^* is proof-theoretically much weaker, as a result of not requiring induction for Σ_1^0 properties. On the other hand, WKL_0^* goes beyond RCA_0 in that WKL implies the existence of noncomputable sets.

It is known that in some important respects WKL_0^* behaves similarly to its stronger cousin $\mathrm{WKL}_0 := \mathrm{RCA}_0 + \mathrm{WKL}$. For example, it was shown in the seminal paper [39] that WKL_0^* is Π_1^1-conservative over RCA_0^*. This is analogous, and can be proved analogously, to the well-known theorem that WKL is Π_1^1-conservative over RCA_0. A more recent observation [10] is that WKL_0^*, like WKL_0, implies the completeness theorem for first-order logic; moreover, it also implies some watered down versions of completeness tailored to cut-free consistency,

which are often helpful as WKL_0^* does not imply cut elimination for first-order logic. This makes it possible to discover numerous connections between the model theory of WKL_0^* and that of first-order arithmetic.

Here, we prove a result that applies specifically to models of WKL_0^* without Σ_1^0-induction and has no apparent analogue for WKL_0. It can be seen as an extension to the second-order setting of some earlier results to the effect that countable models satisfying Σ_n-collection (and exp) but not Σ_n-induction have (in fact, many) nontrivial automorphisms.

Theorem 2.1 (abbreviated version). *Let $(M, \mathcal{X})$ and $(M, \mathcal{Y})$ be countable models of WKL_0^* such that $(M, \mathcal{X} \cap \mathcal{Y}) \models \neg\mathrm{I}\Sigma_1^0$. Then $(M, \mathcal{X})$ and $(M, \mathcal{Y})$ are isomorphic.*

One possible interpretation of the theorem is that the only new sets that can be added to a model of $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma_1^0$ are paths through binary trees. To see this, note that if $(M, \mathcal{X})$ is a countable model of $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma_1^0$, then by [39] it can be ω-extended (that is, extended without changing the first-order universe M) to a model $(M, \mathcal{Y})$ of WKL_0^*. If $G \subseteq M$ is an arbitrary set contained in any other ω-extension of $(M, \mathcal{X})$ satisfying RCA_0^*, then even though $\mathcal{Y}$ might not contain G itself, by Theorem 2.1 it will contain a set H such that (M, G) and (M, H) are isomorphic.

In its more general form, Theorem 2.1 allows the isomorphism to fix a given finite tuple of first- and second-order elements. This has a number of consequences both for WKL_0^* and for other theories.

For example, it follows from Theorem 2.1 that provably in $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma_1^0$ the analytic hierarchy collapses down to Δ_1^1, and even to a slightly more restricted class. If one views arithmetical formulas (i.e., those without second-order quantifiers) as quantifier-free, this means that $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma_1^0$ is a model complete theory, and it is the model companion of $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma_1^0$. As a result, WKL is the strongest Π_2^1 statement that is Π_1^1-conservative over $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma_1^0$. The model completeness phenomenon is quite unexpected, as it does not typically occur in any recognizable form among fragments of first- or second-order arithmetic.

When $n \geqslant 1$ is arbitrary, and $(M, \mathcal{X})$ is a countable model of RCA_0^* satisfying the Σ_n^0-collection scheme $\mathrm{B}\Sigma_n^0$ and a relativization of WKL to Δ_n^0-definable sets but not satisfying $\mathrm{I}\Sigma^0_n$, then the family of Δ^0_n-definable sets of $(M, \mathcal{X})$ forms a model of $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$. Thus, Theorem 2.1 applies to this family, implying that the variety of ω-extensions of models of $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ is also somewhat limited, though in general not quite as drastically as for $n = 1$.

A corollary of this is related to a question of Towsner [42]. Towsner proved that the set of Π^1_2 sentences that are Π^1_1-conservative over $\mathrm{RCA}^*_0 + \mathrm{I}\Sigma^0_n$ is Π_2-complete for each $n \geqslant 1$, and he asked whether this still holds true if $\mathrm{I}\Sigma^0_n$ is replaced by $\mathrm{B}\Sigma^0_n$. We show that this is, rather surprisingly, not the case if additionally one explicitly negates $\mathrm{I}\Sigma^0_n$: for $n \geqslant 1$, the set of Π^1_2 sentences that are Π^1_1-conservative over $\mathrm{RCA}^*_0+\mathrm{B}\Sigma^0_n+\neg\mathrm{I}\Sigma^0_n$ is computably enumerable. On the other hand, we show that Towsner's question as originally stated has a positive answer. The argument for this does not rely on Theorem 2.1, although one of its ingredients is a result (Theorem 6.5) whose known proofs are also based on automorphisms of models of arithmetic.

One further consequence of Theorem 2.1 is connected to Ramsey's theorem for pairs and two colours, RT^2_2, and more precisely to the problem whether $\mathrm{B}\Sigma^0_2$ axiomatizes the Π^1_1 consequences of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$. We prove that this is the case if and only if $\mathrm{B}\Sigma^0_2$ proves all the $\forall\Pi^0_5$ consequences of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ (note that this is known for $\forall\Pi^0_3$ consequences and open from $\forall\Pi^0_4$ onwards); moreover, if this is the case then it can be proved using the "single jump control" method of [4]. We prove these facts by applying Theorem 2.1 to the Δ^0_2-definable sets in models of $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2 + \neg\mathrm{I}\Sigma^0_2$. A different proof is possible using the fact that RT^2_2 is a so-called restricted Π^1_2 formula.

The remainder of this paper is structured as follows. We present the necessary definitions and background in the preliminary Section 1. Section 2 contains the proof of our isomorphism theorem. In Sections 3 and 4, we discuss the consequences of the theorem for WKL^*_0 and for higher levels of the arithmetic hierarchy, respectively. Section 5 concerns implications of the theorem for RT^2_2, as well as a more general discussion of the behaviour of restricted Σ^1_1 formulas under negated Σ^0_1-induction. Finally, in Section 6 we present our positive solution to Towsner's problem.

1 Preliminaries

We assume that the reader has some familiarity with fragments of second-order arithmetic and with models of first- and second-order arithmetic (see [37] or [18] for second-order arithmetic and [23] for models of first-order arithmetic).

We write Δ^0_n, Σ^0_n, Π^0_n to denote the usual formula classes defined in terms of first-order quantifier alternations, but allowing second-order free variables. On the other hand, notation without the superscript 0, like Δ_n, Σ_n, Π_n, represents analogously defined classes of purely first-order, or "lightface", formulas, that do not contain any second-order variables at all. If we want to specify the second-order parameters appearing in a Σ^0_n formula, we use notation like $\Sigma_n(A)$. We extend these conventions to naming theories.

If Γ is a class of formulas, then the class $\forall\exists\Gamma$ contains formulas that consist of a block of universal (first- and/or second-order) quantifiers, followed by a block of existential quantifiers, followed by a formula from Γ. The class $\forall\Gamma$ is defined analogously. For example, $\forall\Sigma^0_n$ and $\forall\Pi^0_{n+1}$ are the same class of Π^1_1 formulas.

The theory RCA_0^*, originally defined in [39], is obtained from RCA_0 by weakening the Σ_1^0-induction axiom $\mathrm{I}\Sigma_1^0$ to Δ_1^0-induction and adding a Π_2 axiom exp that explicitly guarantees the totality of exponentiation. The theory WKL_0^* is obtained from WKL_0 in an analogous way; put differently, WKL_0^* is RCA_0^* plus Weak König's Lemma WKL. Already RCA_0^* proves the collection scheme $\mathrm{B}\Sigma_1^0$, and the first-order consequences of RCA_0^* and of WKL_0^* are axiomatized by $\mathrm{B}\Sigma_1 + \mathrm{exp}$.

Already $\mathrm{I}\Delta_0^0 + \mathrm{exp}$ is strong enough to support a well-behaved universal $\Sigma_n(Y)$ formula $\mathrm{Sat}_n(x, y, Y)$ ("the Σ_n^0 formula (with Gödel number) x with one first- and one second-order variable holds of the number y and the set Y"), for each standard natural number n. As a consequence, we have access to these universal formulas in any theory containing RCA_0^*.

When we consider a model $(M, \mathcal{X})$ of some fragment of second-order arithmetic (or simply work inside this fragment without reference to a specific model), the word *set* without any qualifier refers to an element of the second-order universe $\mathcal{X}$. In contrast, a Σ_n^0*-definable set* (or Σ_n^0*-set* for brevity) is any subset of the first-order universe M that is definable in $(M, \mathcal{X})$ by Σ_n^0 formula. A Δ_n^0*-definable set* or Δ_n^0*-set* is a Σ_n^0-set that is simultaneously definable by a Π_n^0 formula. The notions of a Σ_n- and Δ_n-set are defined in the obvious way.

In general, the models we study only satisfy Δ_1^0-comprehension, so Δ_n^0-sets for $n \geqslant 2$ and Σ_n^0-sets for $n \geqslant 1$ will not always be sets. However, thanks to the availability of universal formulas, we can quantify over Δ_n^0- or over Σ_n^0-sets using second-order quantifiers (e.g. "for every Y and every equivalent pair of a $\Sigma_n(Y)$ and a $\Pi_n(Y)$ formula..."). Whenever we present reasoning or statements that are to be understood as formalized in second-order arithmetic, we indicate quantification over definable sets that might not be sets by putting a tilde over the quantified variable, as in $\widetilde{X}$. So, for example, "for every Σ_1^0-set $\widetilde{X}$ there exists X such that $\forall k\, (k \in \widetilde{X} \leftrightarrow k \in X)$" would be a slightly unusual way of expressing the Σ_1^0-comprehension axiom of ACA_0. Sometimes, in the context of a model-theoretic argument, we may also use a tilde to warn the reader that a given subset of M might not belong to the second-order universe at hand. However, in those situations we treat adding the tilde as a matter of convenience, and we do not strive for consistency (partly because it would be unattainable: often we work with several models at the same time, and what is a set in one model may not be a set in another).

We write $\Delta_n^0\text{-}\mathrm{Def}(M, \mathcal{X})$ for the collection of all the Δ_n^0-sets of $(M, \mathcal{X})$. If A is a subset of M, we write $\Delta_n^0\text{-}\mathrm{Def}(M, A)$ for the collection of all the $\Delta_n(A)$-sets. If $(M, \mathcal{X}) \models \mathrm{B}\Sigma_n^0 + \mathrm{exp}$ where $n \geqslant 1$, then $(M, \Delta_n^0\text{-}\mathrm{Def}(M, \mathcal{X})) \models \mathrm{RCA}_0^*$. A model of RCA_0^* is *topped* if it has the form $(M, \Delta_1^0\text{-}\mathrm{Def}(M, A))$ for some $A \subseteq M$.

The notation $A \leqslant_{\mathrm{T}} B$ means that A is $\Delta_1(B)$-definable. The *join* of A and B, denoted by $A \oplus B$, is $\{\langle 0, k\rangle : k \in A\} \cup \{\langle 1, k\rangle : k \in B\}$; note that $A \leqslant_{\mathrm{T}} A \oplus B$ and $B \leqslant_{\mathrm{T}} A \oplus B$, but at the same time $A \oplus B$ is $\Delta_1(A, B)$-definable. $A \equiv_{\mathrm{T}} B$ means that $A \leqslant_{\mathrm{T}} B$ and $B \leqslant_{\mathrm{T}} A$. If A is a subset of some model M and $(M, A) \models \mathrm{I}\Delta_0(A) + \mathrm{exp}$, then $A^{(n)}$, the *n-th jump* of A, is the $\Sigma_n(A)$-set $\{\langle e, k\rangle : (M, A) \models \mathrm{Sat}_n(e, k, A)\}$. As usual, we write A' for $A^{(1)}$, the *jump* of A. The following result provides a variant of the important method of "jump inversion" adapted to the setting of fragments of second-order arithmetic.

Theorem 1.1 (Belanger [3])**.** *Let $n \geqslant 1$. Assume that M is a structure and $A, C \subseteq M$ are such that $(M, A) \models \mathrm{B}\Sigma_{n+1}^0$ and $(M, A' \oplus C) \models \mathrm{B}\Sigma_n^0$. Then there* *exists* $B \subseteq M$ *such that* C *is* $\Delta_2(A \oplus B)$*-definable and* $(M, A \oplus B) \models \mathrm{B}\Sigma^0_{n+1}$.

A *cut* in a model of arithmetic M is any subset $I \subseteq M$ that contains 0 and is closed downwards and under successor. We can write $I \subseteq_{\mathrm{e}} M$ to indicate that I is a cut in M. Note that if $(M, \mathcal{X}) \models \mathrm{RCA}^*_0$, then no proper cut in M can be a member of $\mathcal{X}$. On the other hand, some proper cut is Σ^0_1-definable in $(M, \mathcal{X})$ if and only if $(M, \mathcal{X}) \models \neg\mathrm{I}\Sigma^0_1$. If I is a Σ^0_1-cut in $(M, \mathcal{X})$, then there is an *unbounded* (i.e. cofinal in M) set $A \in \mathcal{X}$ that can be enumerated in $\mathcal{X}$ in increasing order as $A = \{a_i : i \in I\}$.

For an element s of a model M, $\mathrm{Ack}(s)$ stands for $\{a \in M : M \models a \in_{\mathrm{Ack}} s\}$, where $\in_{\mathrm{Ack}}$ is the usual Ackermann interpretation of set theory in arithmetic ("the a-th bit in the binary notation for s is 1"). In a model $(M, \mathcal{X})$ of RCA^*_0, sets of the form $\mathrm{Ack}(s)$ are exactly the bounded subsets of M that belong to $\mathcal{X}$. The following theorem states an important basic fact about such sets.

Theorem 1.2 (Chong–Mourad [5, Proposition 4])**.** *Let* $(M, \mathcal{X}) \models \mathrm{RCA}^*_0$. *Then for every pair of bounded disjoint* Σ^0_1*-definable sets* $X, Y \subseteq M$ *there exists* $s \in M$ *such that* $\mathrm{Ack}(s) \cap (X \cup Y) = X$.

We use the symbol ω to denote the set of standard natural numbers and the symbol $\mathbb{N}$ to denote the set of natural numbers as formalized within RCA^*_0. In other words, if $(M, \mathcal{X})$ is a model of RCA^*_0, then $\mathbb{N}^{(M,\mathcal{X})}$ is simply the first-order universe M. This convention clashes with the custom of referring to a model that has the same first-order universe as some other model but a smaller second-order universe as an ω-submodel. Because of this, we use the term $\mathbb{\omega}$*-submodel* (and, conversely, $\mathbb{\omega}$*-extension*) in such situations.

A structure $\mathfrak{A}$ is *recursively saturated* if for any computable set of formulas $\{\psi_n(\overline{x}, \overline{y}) : n \in \omega\}$ and any tuple $\overline{b}$ of the appropriate length, if $\mathfrak{A} \models \exists \overline{x} \bigwedge_{i=0}^{n} \psi_i(\overline{x}, \overline{b})$ for each $n \in \omega$, then there is a tuple $\bar{a}$ such that $\mathfrak{A} \models \psi_n(\overline{a}, \overline{b})$ for all n. Recursively saturated structures only play a very minor role in this paper; for more on them, see e.g. [23, Chapters 11.2 & 15].

2 The isomorphism theorem

In this section, we state and prove our main theorem on isomorphisms between models of $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$. One can view the theorem as a generalization to second-order arithmetic of a result of Kossak's [30, Theorem 3.1] (see [31] or [22] for a correction to the proof) saying that every countable model of $\mathrm{B}\Sigma_1(A) + \exp + \neg\mathrm{I}\Sigma_1(A)$ has continuum many automorphisms. In fact, our proof is somewhat reminiscent of Kossak's argument, in which one also finds a truth-coding trick that goes back to Smoryński [40, Lemma 1.2], Kotlarski [34, Lemma 4.4], and Alena Vencovská [unpublished].

Theorem 2.1. *Let* $(M, \mathcal{X})$ *and* $(M, \mathcal{Y})$ *be countable models of* WKL^*_0 *such that* $(M, \mathcal{X} \cap \mathcal{Y}) \models \neg\mathrm{I}\Sigma^0_1$. *Let* $\bar{c}$ *be a tuple of elements of* M *and* $\overline{C}$ *be a tuple of elements of* $\mathcal{X} \cap \mathcal{Y}$. *Then there exists an isomorphism* h *between* $(M, \mathcal{X})$ *and* $(M, \mathcal{Y})$ *such that* $h(\bar{c}) = \bar{c}$ *and* $h(\overline{C}) = \overline{C}$.

Proof. Since $(M, \mathcal{X} \cap \mathcal{Y}) \models \neg\mathrm{I}\Sigma^0_1$, there exists a set $A \in \mathcal{X} \cap \mathcal{Y}$ cofinal in M and a proper cut $I \subsetneq_{\mathrm{e}} M$ such that A can be enumerated in increasing order as $A = \{a_i : i \in I\}$. We may assume w.l.o.g. that I is closed under exp (see

[15, Theorem 2.4] or [28, Lemma 9]) and that the tuple of sets $\overline{C}$ contains both A itself and the $\Delta_1(A)$-definable set $\{\langle i, a_i\rangle : i \in I\}$, so that the relation "$x = a_i$" between x and i is $\Delta_0(\overline{C})$-definable. Finally, by adding a sufficiently large number to all the elements of A if necessary, we may assume that $a_0 > I$.

We build the isomorphism h by a back-and-forth construction. At each step of the construction, we have finite tuples $\overline{r}, \overline{s}$, both in M, and $\overline{R}, \overline{S}$, in $\mathcal{X}, \mathcal{Y}$ respectively, such that we have committed to $h(\overline{r}) = \overline{s}$ and $h(\overline{R}) = \overline{S}$. Initially, $\overline{r} = \overline{s} = \overline{c}$ and $\overline{R} = \overline{S} = \overline{C}$. In each step, we add either a first- or a second-order element to either the domain or the range of h, so that after ω steps of the construction h becomes a bijection from $(M, \mathcal{X})$ onto $(M, \mathcal{Y})$. The inductive condition we maintain is:

(#) there exist $b > I$ and $\epsilon > \omega$ in M such that for each $i \in I$, each $j < b$, and each Δ_0 formula δ (with Gödel number) $< \epsilon$, $(M, \mathcal{X}) \models \delta(a_i, j, \overline{r}, \overline{R})$ iff $(M, \mathcal{Y}) \models \delta(a_i, j, \overline{s}, \overline{S})$.

Of course, the statement that the (potentially nonstandard) formula δ is satisfied in a model is expressed using a fixed truth definition for Δ_0 formulas, Sat_{Δ_0}. Note that (#) holds at the beginning of the construction with $\langle \overline{r}, \overline{R}\rangle = \langle \overline{s}, \overline{S}\rangle = \langle \overline{c}, \overline{C}\rangle$. We have to verify that it can be preserved at each step of the construction. There are two kinds of step to consider: adding a first-order element to our tuple and adding a second-order element.

First-order step. Let us consider the case where we want to add a new first-order element r^* to the domain of our map h, so we need to find $s^* \in M$ such that the tuples $\overline{r}, r^*, \overline{R}$ and $\overline{s}, s^*, \overline{S}$ still satisfy (#). The construction for adding a first-order element to the range of h is analogous.

By the inductive assumption, we have $b > I$ and $\epsilon > \omega$ witnessing (#) for $\overline{r}, \overline{R}, \overline{s}, \overline{S}$. Define $b' = \log\log\log b$ and $\epsilon' = \log\log\epsilon$. Notice that $b' > I$ and $\epsilon' > \omega$, since both I and ω are closed under exp.

Let D stand for the following definable subset of $(I \times b' \times \epsilon')$:

$$\{\langle i, j, \ulcorner\delta\urcorner\rangle : i \in I, j < b', \ulcorner\delta\urcorner < \epsilon' \text{ is (the Gödel number of) a } \Delta_0 \text{ formula,} \\ \text{and } (M, \mathcal{X}) \models \delta(a_i, j, \overline{r}, r^*, \overline{R}))\}$$

Both D and $(I \times b' \times \epsilon') \setminus D$ are bounded and Σ^0_1-definable, so by Theorem 1.2 there exists $j^* \in M$ coding a set $\mathrm{Ack}(j^*)$ such that $\mathrm{Ack}(j^*) \cap (I \times b' \times \epsilon') = D$. Moreover, since $I < b' \ll \log\log b$ and since we can choose ϵ small enough, we can assume that $j^* < \log b$.

Let $i^* \in I$ be such that $r^* \leqslant a_{i^*}$. By the definition of j^*, for each $i \in I$ the structure $(M, \mathcal{X})$ satisfies:

$$\exists y \leqslant a_{i^*}\, \forall i' \leqslant i\, \forall j \leqslant b' \bigwedge_{\ulcorner\delta\urcorner < \epsilon'} \big(\delta(a_{i'}, j, \overline{r}, y, \overline{R}) \leftrightarrow \langle i', j, \ulcorner\delta\urcorner\rangle \in_{\mathrm{Ack}} j^*\big), \tag{1}$$

as witnessed by $y := r^*$. If $i \in I$ and $i \geqslant i^*$, then (1) implies:

$$\exists x, y \leqslant a_i \Bigg(x = a_{i^*} \wedge y \leqslant x \wedge \forall i' \leqslant i\, \forall v \leqslant a_i \Big(v = a_{i'} \rightarrow \\ \rightarrow \forall j \leqslant b' \bigwedge_{\ulcorner\delta\urcorner < \epsilon'} \big(\delta(v, j, \overline{r}, y, \overline{R}) \leftrightarrow \langle i', j, \ulcorner\delta\urcorner\rangle \in_{\mathrm{Ack}} j^*\big)\Big)\Bigg). \tag{2}$$

For each fixed $i \in I$, the statement (2) can be expressed as a (nonstandard) Δ_0 formula γ with parameters $i^*, b', j^*, a_i, \overline{r}, \overline{R}$. The formula γ consists of a fixed-length part independent of ϵ', followed by a conjunction whose conjuncts correspond to Δ_0 formulas δ with $\ulcorner\delta\urcorner < \epsilon'$. Each such conjunct is an equivalence, with the formula δ, which consists of at most $\log \epsilon'$ symbols, written out on the left-hand side and with $\ulcorner\delta\urcorner$ referred to by its (say binary) numeral on the right-hand side. Thus, in total, γ has length $O(\epsilon' \cdot \log(\epsilon'))$, which means that $\ulcorner\gamma\urcorner < \epsilon$. Moreover, the tuple of parameters $\langle i^*, b', j^*\rangle$ is smaller than b.

We can therefore apply (#) from the inductive hypothesis with $\delta := \gamma$ and $j := \langle i^*, b', j^*\rangle$ in order to conclude that for each sufficiently large $i \in I$, the structure $(M, \mathcal{Y})$ satisfies (2) with $\overline{r}, \overline{R}$ replaced by $\overline{s}, \overline{S}$. From this it easily follows that $(M, \mathcal{Y})$ in fact satisfies

$$\exists y \leqslant a_{i^*}\, \forall i' \leqslant i\, \forall j \leqslant b' \bigwedge_{\ulcorner\delta\urcorner < \epsilon'} \left(\delta(a_{i'}, j, \overline{s}, y, \overline{S}) \leftrightarrow \langle i', j, \ulcorner\delta\urcorner\rangle \in \mathrm{Ack}(j^*)\right) \tag{3}$$

for each $i \in I$. Note that (3) is (1) with $\overline{r}, \overline{R}$ replaced by $\overline{s}, \overline{S}$.

By $\mathrm{B}\Sigma^0_1$ in $(M, \mathcal{Y})$, there must exist some $y \leqslant a_{i^*}$ that witnesses (3) for all $i \in I$. We can choose any such y as our s^*.

Second-order step. This step is somewhat similar to the first-order one, with the role of $\mathrm{B}\Sigma^0_1$ now played by WKL. As before, we consider only the case where we want to add a new second-order element $R^* \in \mathcal{X}$ to the domain of h and we need to find $S^* \in \mathcal{Y}$ such that $\overline{r}, \overline{R}, R^*$ and $\overline{s}, \overline{S}, S^*$ still satisfy (#).

By the inductive assumption, we have $b > I$ and $\epsilon > \omega$ witnessing (#) for $\overline{r}, \overline{R}, \overline{s}, \overline{S}$. The parameters b' and ϵ' are defined as in the first-order step. The set D and the element j^* are also defined as before, but with $\overline{r}, r^*, \overline{R}$ replaced by $\overline{r}, \overline{R}, R^*$.

It follows from the definition of j^* that $(M, \mathcal{X})$ satisfies

$$\begin{aligned}\exists F \subseteq [0, \log a_i)\, \forall i' \leqslant i\, \forall v \leqslant \log\log a_i \Big(& v = a_{i'} \rightarrow \\ & \rightarrow \forall j \leqslant b' \bigwedge_{\ulcorner\delta\urcorner < \epsilon'} \left(\delta(v, j, \overline{r}, \overline{R}, F) \leftrightarrow \langle i', j, \ulcorner\delta\urcorner\rangle \in_{\mathrm{Ack}} j^*\right)\Big)\end{aligned} \tag{4}$$

for each large enough $i \in I$, as witnessed by $F := R^* \cap [0, \log a_i)$. (The reason for the restriction to large $i \in I$ is that we want all terms in a Δ_0 formula δ with $\ulcorner\delta\urcorner < \epsilon'$ to evaluate to a number below $\log a_i$ on arguments below $\max(\log\log a_i, b', \max(\overline{r}))$. In this way, the value of δ on such arguments is unchanged when we replace R^* by $R^* \cap [0, \log a_i)$.)

Arguing as in the first-order case, one can check that for a fixed $i \in I$ the statement (4) is equivalent to a Δ_0 formula with Gödel number below ϵ and parameters $b', j^*, a_i, \overline{r}, \overline{R}$, where the tuple $\langle b', j^*\rangle$ is below b. Therefore, we can apply (#) from the inductive hypothesis to conclude that $(M, \mathcal{Y})$ satisfies

$$\begin{aligned}\exists F \subseteq [0, \log a_i)\, \forall i' \leqslant i\, \forall v \leqslant \log\log a_i \Big(& v = a_{i'} \rightarrow \\ & \rightarrow \forall j \leqslant b' \bigwedge_{\ulcorner\delta\urcorner < \epsilon'} \left(\delta(v, j, \overline{s}, \overline{S}, F) \leftrightarrow \langle i', j, \ulcorner\delta\urcorner\rangle \in_{\mathrm{Ack}} j^*\right)\Big)\end{aligned} \tag{5}$$

for each large enough $i \in I$. Note that (5) is (4) with $\overline{r}, \overline{R}$ replaced by $\overline{s}, \overline{S}$.

Let $T \in \mathcal{Y}$ be the tree consisting of all 0–1 strings σ such that, for each $i \in I$ large enough (the largeness is expressed by a single inequality, with no quantifiers involved) and such that $a_i < |\sigma|$, the string $\sigma\restriction_{\log a_i}$ is the characteristic function of $F \subseteq [0, \log a_i)$ satisfying (5) for i. By the previous paragraph, there are arbitrarily large elements of T, so by WKL in $(M, \mathcal{Y})$, there is an infinite path $B \in \mathcal{Y}$ through T. Any such path is the characteristic function of a set that we can use as S^*. ☐

Remark. A more refined version of the proof of Theorem 2.1, in which the exponentially closed cut used to lower-bound the parameter b is decoupled from the Σ^0_1-cut indexing a cofinal subset of the first-order universe, can be used to show the following. If (M, A) is a countable model of $\mathrm{B}\Sigma_1(A) + \exp + \neg\mathrm{I}\Sigma_1(A)$, then any exponentially closed cut in M that contains some $\Sigma_1(A)$-definable cut is the greatest initial segment fixed pointwise by some automorphism of (M, A). This is also the information about greatest pointwise fixed initial segments that can be obtained from the previously published arguments drawing on [40] such as those of [30]/[31] or [22].

We record the following immediate consequence of Theorem 2.1. Further implications of the theorem for WKL^*_0 are discussed in the next section.

Corollary 2.2. *Let $(M, \mathcal{X})$, $(M, \mathcal{Y})$ be models of WKL^*_0 such that $(M, \mathcal{X} \cap \mathcal{Y}) \models \neg\mathrm{I}\Sigma^0_1$. Let $\bar{c}$ be a tuple of elements of M and $\overline{C}$ be a tuple of elements of $\mathcal{X} \cap \mathcal{Y}$.*

Then $(M, \mathcal{X}, \bar{c}, \overline{C}) \equiv (M, \mathcal{Y}, \bar{c}, \overline{C})$, and if $\mathcal{X} \subseteq \mathcal{Y}$ then $(M, \mathcal{X}) \preccurlyeq (M, \mathcal{Y})$, where the elementarity applies to all $\mathcal{L}_2$-formulas.

3 Consequences for WKL^*_0

In this section, we show how Theorem 2.1 implies that Weak König's Lemma combined with the failure of Σ^0_1-induction has a number of unusual properties. In particular, we prove that in $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ the analytic hierarchy collapses (Corollary 3.3, Theorem 3.13) and the low basis theorem fails in a very strong sense (Theorem 3.9). We also show that WKL is the strongest Π^1_2 statement that is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ (Theorem 3.6).

Most of the main results of this section can be interpreted in general model-theoretic terms, cf. e.g. [41, Chapter 2.2]. If we view arithmetical formulas as quantifier-free, Π^1_1 as purely universal, Π^1_2 as $\forall\exists$, and so forth, then, for instance, Corollary 3.3 says that the $\forall\exists$ theory $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ is model complete. Together with the Π^1_1 conservativity of $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ over $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ from Simpson–Smith [39], this implies that $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ is the model companion of $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ (by Theorem 3.13, it actually has the slightly stronger property of being a model completion of $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$). Thus, models of $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ are precisely the existentially closed models of $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$, and $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ is the strongest $\forall\exists$-theory that has the same purely universal consequences as $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ (this is essentially what Theorem 3.6 tells us).

In fact, most of the results mentioned above could be proved using Corollary 2.2 and general model-theoretic arguments. However, we give proofs that are more specialized to arithmetic, because they provide some additional information. We begin by checking that Σ^0_1-induction is not needed to justify the well-known fact that a model of WKL contains many coded submodels of WKL [37, Corollary VIII.2.7].

Definition 3.1. Given $(M, \mathcal{X}) \models \mathrm{RCA}_0^*$, a set $W \in \mathcal{X}$, and an element $k \in M$, let W_k stand for $\{j \in M : \langle k, j\rangle \in W\}$. We say that W *codes* the family of sets $\{W_k : k \in M\}$, and we refer to $(M, \{W_k : k \in M\})$ as a *coded* ω*-submodel of* $(M, \mathcal{X})$. We use the abbreviated term *coded* ω*-model* if $(M, \mathcal{X})$ is clear from the context or irrelevant, and also to refer to $(\mathbb{N}, \{W_k : k \in \mathbb{N}\})$ when working in a formal theory.

Lemma 3.2. *Let* $(M, \mathcal{X}) \models \mathrm{RCA}_0^*$ *and let* $A \in \mathcal{X}$. *There exists a* $\Delta_1(A)$*-definable infinite 0–1 tree* $T \in \mathcal{X}$ *such that if* $W \subseteq M$ *is an infinite path in* T*, then* $(M, \{W_k : k \in M\})$ *is a model of* WKL_0^* *with* $A = W_0$.

As a consequence, for every $(M, \mathcal{X}) \models \mathrm{WKL}_0^*$ *and every* $A \in \mathcal{X}$*, there exists* $W \in \mathcal{X}$ *coding an* ω*-model* $(M, \mathcal{W})$ *of* WKL_0^* *with* $A \in \mathcal{W}$.

Note that it is not assumed in the lemma that $W \in \mathcal{X}$ or even that $\mathrm{B}\Sigma_1(W)$ holds: an arbitrary subset $W \subseteq M$ with the property that the characteristic function of $W\restriction_k$ is a node of T for every $k \in M$ will have the property that $(M, \{W_k : k \in M\}) \models \mathrm{WKL}_0^*$.

Proof. Let $(M, \mathcal{X}) \models \mathrm{RCA}_0^*$ and $A \in \mathcal{X}$ be given.

It is easy to check that the well-known equivalence between WKL and the Σ_1^0-separation principle [37, Lemma IV.4.4] holds over RCA_0^*. Moreover, Σ_1^0-separation clearly implies Δ_1^0-comprehension. Thus, given $W \subseteq M$, to prove that the structure $(M, \{W_k : k \in M\})$ is a model of WKL_0^* it suffices to show that it satisfies $\mathrm{I}\Delta_0^0$, that $\{W_k : k \in M\}$ is closed under join, and that for any $k \in M$ and any two $\Sigma_1(X)$ formulas $\varphi(x, X), \psi(x, X)$ (possibly with first-order parameters from M), if $\varphi(\cdot, W_k)$ and $\psi(\cdot, W_k)$ define disjoint sets, then there is some W_ℓ separating them. Given an arbitrary tree from a model $(M, \mathcal{X}) \models \mathrm{RCA}_0^*$ and an arbitrary $W \subseteq M$ that is a path in that tree, $\mathrm{I}\Delta_0(W)$ will always hold, and a fortiori $\{W_k : k \in M\}$ will satisfy $\mathrm{I}\Delta_0^0$, simply because bounded initial segments of W will always have the form $\mathrm{Ack}(s)$ for some $s \in M$. So, it is enough to verify closure under join and Σ_1^0-separation.

Let $(\varphi_i(x, X))_{i \in M}$ be an effective listing of all Σ_1^0 formulas in M with a unique first-order variable x and a unique second-order free variable X. (The formulas may involve numerals that represent first-order parameters.) We take T to be a 0–1 tree describing finite approximations to some W such that:

(a) $W_0 = A$,

(b) $W_{\langle 0,i,j\rangle} = W_i \oplus W_j$ for every i, j,

(c) for every $i, j, k \in M$, if the sets defined by $\varphi_i(\cdot, W_k)$ and $\varphi_j(\cdot, W_k)$ are disjoint, then $W_{\langle 1,i,j,k\rangle}$ is a separating set for them.

Formally, to determine whether a finite binary string σ belongs to T, we look at the largest s such that $\langle i, j\rangle < |\sigma|$ for each $i, j < s$, so that σ can be viewed as containing an $s \times s$ binary-valued matrix. The conditions (a)–(c) are then expressed as requirements concerning this matrix. For instance, (a) is expressed by requiring that $\sigma(\langle 0, i\rangle) = 1$ iff $i \in A$, for each $i < s$. We leave (b) to the reader. The condition (c) is expressed by requiring that, whenever $\langle 1, i, j, k\rangle < s$ and there is no $x < s$ such that there are witnesses below s for both the formulas

$$\begin{aligned}
\psi_i(x) &:= \varphi_i(x, \{w < s : \sigma(\langle k, w\rangle) = 1\}),\\
\psi_j(x) &:= \varphi_j(x, \{w < s : \sigma(\langle k, w\rangle) = 1\}),
\end{aligned}$$

then for each $x < s$, the existence of $y < s$ witnessing $\psi_i(x)$ implies that $\sigma(\langle\langle 1,i,j,k\rangle,x\rangle) = 1$, and the existence of $y < s$ witnessing $\psi_j(x)$ implies that $\sigma(\langle\langle 1,i,j,k\rangle,x\rangle) = 0$.

It is straightforward to verify that T is an infinite $\Delta_1(A)$-definable binary tree. By our discussion above, if $W \subseteq M$ is an infinite path in T, then W codes an ω-model satisfying WKL_0^* and containing A. $\square$

Remark. The construction of a single infinite binary tree whose paths correspond to coded models of WKL has been used earlier, for instance in the context of comparing variants of Weihrauch reducibility [19, Proposition 4.9]. It can also be applied to give a relatively simple proof of the result of [15, 1] that WKL_0 does not have superpolynomial proof speedup over RCA_0 for Π^1_1 sentences (see [2], or see [43] for a very similar argument expressed in terms of the arithmetized completeness theorem). Combined with the formalized forcing argument described at the end of the present section, it can also prove the fact, alluded to in [27], that a similar non-speedup result holds also for WKL_0^* over RCA_0^*.

Note that for any $\mathcal{L}_2$-formula $\varphi(\overline{x},\overline{X})$, there is a single arithmetical formula in variables $\overline{x},\overline{X},W$ that expresses the property "W codes an ω-model containing $\overline{X}$ and satisfying $\varphi(\overline{x},\overline{X})$" in RCA_0^*. The number of first-order quantifier alternations in this formula will depend on φ.

Corollary 3.3. *Over* $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1$*, any* $\mathcal{L}_2$*-formula* $\varphi(\overline{x},\overline{X})$ *is provably equivalent both to a* Σ^1_1 *formula and to a* Π^1_1 *formula.*

More specifically, $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1$ *proves that the following three statements are equivalent for any* $\overline{x},\overline{X}$*:*

(i) $\varphi(\overline{x},\overline{X})$*,*

(ii) "there exists a coded ω*-model of* $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1 + \varphi(\overline{x},\overline{X})$*",*

(iii) "every coded ω*-model of* $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1$ *containing* $\overline{x},\overline{X}$ *satisfies* $\varphi(\overline{x},\overline{X})$*".*

Proof. By Lemma 3.2, it is provable in $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1$ that for every $\overline{x},\overline{X}$ there is a coded ω-model of $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1$ containing $\overline{X}$. By Corollary 2.2, it is provable in WKL_0^* that any such model satisfies $\varphi(\overline{x},\overline{X})$ if and only if $\varphi(\overline{x},\overline{X})$ holds. This proves both (i) $\leftrightarrow$ (ii) and (i) $\leftrightarrow$ (iii). $\square$

Corollary 3.3 says that in $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1$ the analytic hierarchy collapses to Δ^1_1. At the end of this section, we will explain how this collapse result can be strengthened.

We can use Corollary 3.3 and other consequences of Theorem 2.1 to give a characterization of Π^1_1-conservativity over $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$ for Π^1_2 sentences. Before that, however, we digress briefly in order to point out that the Π^1_1 consequences of $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$ are not as pathological a theory as might be supposed.

Proposition 3.4. *The* Π^1_1 *consequences of* $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$ *are contained in those of* ACA_0*; in particular, they are true in all* ω*-models. They are incomparable to the* Π^1_1 *consequences of* $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n$*, for any* $n \in \omega$*.*

Proof. As discussed in Section 6, the principle $\mathrm{C}\Sigma^0_{n+1}$ is a Π^1_1 statement provable in $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$ but not in $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n$. Of course, $\mathrm{Con}(\mathrm{I}\Delta_0 + \exp)$ or the totality of the iterated exponential function are examples of Π^1_1 statements provable in RCA_0 but not in $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$.

It remains to prove that the Π^1_1 consequences of $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ are contained in those of ACA_0, or by contraposition, that any Σ^1_1 statement consistent with ACA_0 is consistent with $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$. This is implicit in the proof of [29, Proposition 2.2]. We spell out the argument. Let $(M, \mathcal{X})$ be a countable recursively saturated model of $\mathrm{ACA}_0 + \exists X\, \alpha(X)$, where α is arithmetical, and let $A \in \mathcal{X}$ be such that $(M, A) \models \alpha(A)$. The one-sorted structure (M, A) is a countable recursively saturated model of $\mathrm{PA}(A)$.

By recursive saturation, for every $a \in M$ there exists $b \in M$ that is above any element definable in (M, A) with parameters below a. By another application of recursive saturation, there exists a sequence $\langle a_k \rangle_{k<c}$ for c nonstandard such that each a_k is above any element definable in (M, A) with parameters below a_{k-1}. Let I be the initial segment of M generated by $C = \{a_n : n \in \omega\}$, and let $A_I = A \cap I$. Note that I is obviously closed under exp and that $A_I \oplus C$ is the intersection with I of a definable set, so the structure $(I, A_I \oplus C)$ satisfies $\mathrm{B}\Sigma^0_1$. Of course, it does not satisfy $\mathrm{I}\Sigma^0_1$, because ω is $\Sigma_1(C)$-definable.

Since I is closed under definability in (M, A) and $\mathrm{PA}(A)$ has definable Skolem functions, (I, A_I) is an elementary substructure of (M, A). Thus, $(I, A_I) \models \alpha(A_I)$. Putting things together, we conclude that $(I, \Delta^0_1\text{-Def}(I, A_I \oplus C))$ is a model of $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1 + \exists X\, \alpha(X)$. □

Lemma 3.5. *Let ψ be a Π^1_2 sentence. Then ψ is Π^1_1-conservative over* $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ *if and only if* RCA^*_0 *proves the Π^1_1 sentence "every coded ω-model of* $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ *satisfies ψ".*

Proof. If ψ is a Π^1_2 sentence which is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$, then, by [39] and a routine union of chains argument (cf. [44]), $\mathrm{WKL} \wedge \psi$ is also Π^1_1-conservative over $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$. By Corollary 3.3, $\mathrm{WKL}^*_0 + \psi + \neg\mathrm{I}\Sigma^0_1$ proves that every coded ω-model of $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ satisfies ψ. So, by Π^1_1-conservativity, also $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ proves this statement, which is then obviously provable in RCA^*_0 since $\mathrm{RCA}^*_0 + \mathrm{I}\Sigma^0_1$ rules out the existence of coded ω-models of $\neg\mathrm{I}\Sigma^0_1$.

On the other hand, if there is a Σ^1_1 sentence ξ consistent with $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ but not with $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1 + \psi$, then by [39] there is a model of $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1 + \xi$. Clearly, this model must also satisfy $\neg\psi$, so by Corollary 3.3 it contains a coded ω-submodel satisfying $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1 + \neg\psi$. □

Remark. Even though Corollary 3.3 applies to arbitrary $\mathcal{L}_2$-formulas φ, the proof of Lemma 3.5 employs a union of chains argument that only works for Π^1_2 statements. In fact, Lemma 3.5 cannot be generalized to Σ^1_2 sentences ψ, as witnessed by $\psi := \neg\mathrm{WKL}$ [28, Proposition 11].

Lemma 3.5 already implies that the set of Π^1_2 sentences that are Π^1_1-conservative over $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ is c.e., and thus it is a computably axiomatized theory. The next result states that in fact this theory has a rather simple axiomatization.

Theorem 3.6. *Let ψ be a Π^1_2 sentence. Then ψ is Π^1_1-conservative over* $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ *if and only if* $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1 \vdash \psi$.

Proof. Of course, if the Π^1_2 sentence ψ follows from $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$, then it is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$, by [39].

On the other hand, if ψ is Π^1_2 and Π^1_1-conservative over $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$, then, by Lemma 3.5, RCA^*_0 proves that every coded ω-model of $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ satisfies ψ. But then Corollary 3.3 implies that $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ proves ψ. □

Remark. Theorem 3.6 means in particular that, in contrast to the tree forcing used to add paths through binary trees to a model, any forcing notion that can be used to obtain witnesses to statements that do not follow from $\mathrm{WKL} + \neg\mathrm{I}\Sigma^0_1$ will not in general preserve $\mathrm{B}\Sigma^0_1$ when applied to a ground model satisfying $\mathrm{B}\Sigma^0_1 + \exp + \neg\mathrm{I}\Sigma^0_1$. Note that unprovability from $\mathrm{WKL} + \neg\mathrm{I}\Sigma^0_1$ is known e.g. for relatively weak Ramsey-theoretic such as the cohesive Ramsey's Theorem CRT^2_2 (cf. [14]), which is very easy to witness using either Cohen or Mathias forcing.

Similar observations specifically about Cohen forcing, but in the context of models of $\neg\exp$, were made in [11] and [12]. Other results showing that it can be impossible to preserve collection while forcing include [38, Theorem 1], [20, Theorem 4.3].

An important feature of WKL is the low basis theorem [21]: if T is any infinite 0–1 tree, then T has an infinite path W which is *low* in T, i.e. every $\Delta_2(T \oplus W)$-definable set is $\Delta_2(T)$-definable. It is known that the low basis theorem holds provably in RCA_0, in the sense that if $(M, \Delta^0_1\text{-Def}(M,T)) \models \mathrm{RCA}_0 \wedge$ "T is an infinite 0–1 tree", then there is some $W \subseteq M$ low in T such that W is an infinite path in T and $(M, \Delta^0_1\text{-Def}(M, T \oplus W))$ still satisfies RCA_0 ([16], see [17, Chapter I.3(b)]).

Chong and Yang [8] showed that some important properties of low sets fail in the absence of Σ^0_1-induction: for example, over $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma_1$ there is no non-computable low Σ_1-set. Here, we prove that the low basis theorem fails in RCA^*_0 in a strong way: in general, a computable tree in a model of RCA^*_0 will not even have an arithmetical path.

Lemma 3.7. *Let $(M, \mathcal{X}) \models \mathrm{RCA}^*_0$, and let $W \in \mathcal{X}$ code an ω-model $(M, \mathcal{W}) \models \mathrm{RCA}^*_0$. Then $W \notin \mathcal{W}$.*

Proof. If W codes $\mathcal{W}$, then the sequence $\langle W_k \rangle_{k \in M}$ contains all sets that belong to $\mathcal{W}$. By a standard diagonalization argument, such a sequence cannot itself belong to $\mathcal{W}$. □

Lemma 3.8. *Let $(M, \mathcal{X}) \models \mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$. Then for every A in $\mathcal{X}$, there is a set $B \in \mathcal{X}$ which is not arithmetically definable in A.*

Proof. Suppose otherwise. By Lemma 3.2 there exists $W \in \mathcal{X}$ coding an ω-model $\mathcal{W}$ of $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ such that $A \in \mathcal{W}$. Assume that there is an arithmetical formula $\varphi(x, A)$, with no set parameters other than A, which defines W in (M, A). Then $(M, \mathcal{X}) \models \exists X\, \forall x\, (x \in X \leftrightarrow \varphi(x, A))$. On the other hand, Lemma 3.7 implies that $W \notin \mathcal{W}$, so $(M, \mathcal{W}) \not\models \exists X\, \forall x\, (x \in X \leftrightarrow \varphi(x, A))$. This contradicts Corollary 2.2. □

Theorem 3.9. *Let $(M, \mathcal{X}) \models \mathrm{RCA}^*_0$ and let $A \in \mathcal{X}$ be such that $(M, A) \models \neg\mathrm{I}\Sigma_1(A)$. Then there exists a $\Delta_1(A)$-definable infinite 0–1 tree T such that for any $W \subseteq M$, if W is an infinite path in T, then W is not arithmetically definable in A (in particular, it is not low in A).*

Proof. Let $(M, \mathcal{X})$ and A be as above, and let T be the $\Delta_1(A)$-definable tree from Lemma 3.2. If W is an infinite path in T, then, letting $\mathcal{W} := \{W_k : k \in M\}$, we have $(M, \mathcal{W}) \models \mathrm{WKL}_0^*$ and $A \in \mathcal{W}$. If W were arithmetically definable in A, then every set $B \in \mathcal{W}$ would also be arithmetically definable in A, contradicting Lemma 3.8. □

We may reformulate this as a reverse mathematics-style statement, which follows immediately from Theorem 3.9 and the aforementioned provability of the low basis theorem in RCA_0.

Corollary 3.10. *For each $n \geqslant 2$, the following statements are equivalent provably in RCA_0^*:*

(i) RCA_0*,*

(ii) for every infinite 0–1 tree T, there exists a $\Sigma_n(T)$-set $\widetilde{W}$ such that $\widetilde{W}$ is an infinite path in T and $\mathrm{I}\Sigma_1(\widetilde{W})$ holds,

(iii) for every infinite 0–1 tree T, there exists a $\Sigma_n(T)$-set $\widetilde{W}$ such that $\widetilde{W}$ is an infinite path in T.

As far as we know, the following purely model-theoretic question remains open. If the answer is negative, then it has to be witnessed by a model that is not only uncountable, but does not have countable cofinality.

Question 3.11. Is it the case that for every model $(M, \mathcal{X}) \models \mathrm{RCA}_0^*$ and every infinite 0–1 tree $T \in \mathcal{X}$, there exists an infinite path in T?

To conclude this section, we return to the topic of the collapse of the analytic hierarchy. We discuss how to strengthen Corollary 3.3 to say that provably in RCA_0^*, if A is any set witnessing the failure of $\mathrm{I}\Sigma_1^0$, then each $\mathcal{L}_2$-formula is equivalent not merely to a Δ_1^1 formula, but to an arithmetical formula with A as parameter. In particular, if even $\mathrm{I}\Sigma_1$ fails, then every $\mathcal{L}_2$-formula is equivalent to an arithmetical one with no additional parameters. The strengthened collapse result is proved by a forcing argument which is routine but tedious, so we only provide a sketch.

It is well-known that if T is an infinite 0–1 tree in a countable model of RCA_0^*, then forcing with $\Delta_1(T)$-definable infinite subtrees of T gives rise to an infinite path in T that still satisfies $\mathrm{B}\Sigma_1^0$ [39]. The analogous forcing construction for adding paths to trees that live in models of RCA_0 was formalized within RCA_0 in [1, Sections 4–5]. Here, we formalize in RCA_0^* a variant of this forcing such that the generic sets correspond to coded ω-models of WKL_0^*.

The following definition is made in RCA_0^*.

Definition 3.12. Given any set X and another set A that we treat as a parameter, let $T_{X,A}$ be a tree defined like the one in Lemma 3.2 but with paths corresponding to sets W that code an ω-model of WKL_0^* with $W_0 = X$, $W_1 = A$. The *conditions* of the forcing notion $\mathbb{P}_{X,A}$ are the $\Delta_1(X, A)$-definable infinite subtrees of $T_{X,A}$, ordered by inclusion. A *first-order name* has the form $\dot{k}$ for a natural number k, and it is intended to denote k itself. A *second-order name* has the form $\dot{G}_k$ for a natural number k, and it is intended to denote G_k where G is a generic path in $T_{X,A}$.

For a condition S and a *sentence* φ of the language obtained by extending $\mathcal{L}_2$ with all first- and second-order names, treated as constants, the forcing relation $S \Vdash_{X,A} \varphi$ (with the subscripts often omitted below) is defined in the following way. If φ is a purely first-order atom, then $S \Vdash \varphi$ if and only if φ holds (under the intended interpretation of names). If φ has the form $t(\bar{\dot{k}}) \in \dot{G}_\ell$, then $S \Vdash \varphi$ if the set $\{\sigma \in S : \sigma(\langle \ell, t(\bar{k})\rangle) = 0\}$ is finite. The $\Vdash$ relation is extended to non-atomic formulas (which we take to be built using $\neg, \wedge$, and $\exists$) as follows:

$$\begin{aligned} S \Vdash \neg\varphi &:= \forall Q \preceq S\, (Q \nVdash \varphi), \\ S \Vdash \varphi \wedge \psi &:= (S \Vdash \varphi \wedge S \Vdash \psi), \\ S \Vdash \exists x\, \varphi &:= \forall Q \preceq S\, \exists R \preceq Q\, \exists k\, (R \Vdash \varphi(\dot{k})), \\ S \Vdash \exists X\, \varphi &:= \forall Q \preceq S\, \exists R \preceq Q\, \exists k\, (R \Vdash \varphi(\dot{G}_k)). \end{aligned}$$

Note that for any $\mathcal{L}_2$-formula $\varphi(x_1, \ldots, x_n, Y_1, \ldots, Y_m)$, the statement $S \Vdash_{X,A} \varphi(\bar{\dot{x}}, \dot{G}_{y_1}, \ldots, \dot{G}_{y_m})$ can be expressed by an arithmetical formula in $\bar{x}, \bar{y}, S, X$, and A.

For a model $(M, \mathcal{X}) \models \mathrm{RCA}_0^*$ and $X, A \in \mathcal{X}$, a generic filter G in $\mathbb{P}_{X,A}$ can be identified with an infinite path in $T_{X,A}$, which codes an ω-model $(M, \{G_k : k \in M\}$ of WKL_0^* with $G_0 = X$ and $G_1 = A$. We refer to this structure as $M[G]$. One can prove the following two statements by (simultaneous) induction on the complexity of φ. Firstly, if S is a condition, then $S \Vdash \varphi$ iff for every generic G with $S \in G$ it holds that $M[G] \models \varphi$ (as usual, under the intended interpretation of names). Secondly, if G is generic and $M[G] \models \varphi$, then there is some condition $S \in G$ such that $S \Vdash \varphi$.

If $(M, \mathcal{X})$ is a model of WKL_0^* and A is a witness for the failure of $\mathrm{I}\Sigma_1^0$ in $(M, \mathcal{X})$, then by Corollary 2.2, for any $\bar{c} \in M$, any $X \in \mathcal{X}$, and G generic for $\mathbb{P}_{X,A}$, the structure $(M[G], \bar{c}, X)$ is elementarily equivalent to $(M, \mathcal{X}, \bar{c}, X)$. This means that either every condition will force $\varphi(\bar{\dot{c}}, \dot{G}_0)$ or every condition will force $\neg\varphi(\bar{\dot{c}}, \dot{G}_0)$, depending on whether $\varphi(\bar{c}, X)$ holds. Thus, we obtain the following collapse result:

Theorem 3.13. *Let $\varphi(\bar{x}, X)$ be an $\mathcal{L}_2$-formula. Then WKL_0^* proves that following statements are equivalent for any $\bar{x}, X$ and any Y such that $\neg\mathrm{I}\Sigma_1(Y)$ holds:*

(i) $\varphi(\bar{x}, X)$,

(ii) $\Vdash_{X,Y} \varphi(\bar{\dot{x}}, \dot{G}_0)$.

In particular, over $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma_1$, $\varphi(\bar{x}, X)$ is equivalent to the arithmetical formula $\Vdash_{X,\emptyset} \varphi(\bar{\dot{x}}, \dot{G}_0)$, which has no free variables other than X.

4 Generalization to higher levels

If $n \geqslant 2$ and $(M, \mathcal{X})$ is a model of $\mathrm{RCA}_0 + \mathrm{B}\Sigma_n^0 + \neg\mathrm{I}\Sigma_n^0$, then the Δ_n^0-definable sets of $(M, \mathcal{X})$ form a model of $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma_1^0$. Thus, it is reasonable to expect that the results of Sections 2 and 3 say something about $\mathrm{RCA}_0 + \mathrm{B}\Sigma_n^0 + \neg\mathrm{I}\Sigma_n^0$, for instance about Π_1^1-conservativity over that theory. The case of $n = 2$ is particularly interesting, as some prominent problems concerning the first-order consequences of Ramsey's Theorem for pairs and related statements boil down to the question whether these statements are Π_1^1-conservative over $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0 + \neg\mathrm{I}\Sigma_2^0$.

In this section, we show that Theorem 2.1 does indeed have some consequences for higher levels of the arithmetic hierarchy. First, however, we generalize the results of Proposition 3.4 to $n \geqslant 2$.

Proposition 4.1. *For each $n \geqslant 1$, the Π^1_1 consequences of* $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ *are contained in those of* ACA_0 *and incomparable to the Π^1_1 consequences of* $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_m$ *for each $m \geqslant n$.*

Proof. The argument for incomparability with the Π^1_1 consequences of $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_m$ for $m \geqslant n$ is essentially the same as in the proof of Proposition 3.4 and uses the statements $\mathrm{C}\Sigma^0_{m+1}$ and $\mathrm{Con}(\mathrm{I}\Sigma_{n-1})$.

The argument showing that the Π^1_1 consequences of $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ are contained in those of ACA_0 is also similar to the one in Proposition 3.4, but now it has to be combined with a "jump inversion" argument based on [3]. Let $(M, \mathcal{X})$ be a countable recursively saturated model of $\mathrm{ACA}_0 + \alpha(A)$, where $A \in \mathcal{X}$ and α is arithmetical, and let C, I, A_I be obtained as in the proof of Proposition 3.4. We then know that $(I, A_I \oplus C) \models \mathrm{B}\Sigma^0_1 + \alpha(A_I)$ and $(I, C) \models \neg\mathrm{I}\Sigma^0_1$. Moreover, (I, A_I) is an elementary substructure of (M, A), which means in particular that (I, A_I) satisfies induction for all arithmetical formulas, and, by construction, that $(I, (A_I)^{(m)} \oplus C) \models \mathrm{B}\Sigma^0_1$ for any m.

Let $C_1 := C$. If $n = 1$, we have nothing more to do. Otherwise, note that $(I, (A_I)^{(n-2)}) \models \mathrm{B}\Sigma^0_2$. Since $(I, (A_I)^{(n-1)} \oplus C_1) \models \mathrm{B}\Sigma^0_1$, we can use Belanger's jump inversion Theorem 1.1 to obtain some $C_2 \subseteq I$ such that $(I, (A_I)^{(n-2)} \oplus C_2) \models \mathrm{B}\Sigma^0_2$ and C_1 is $\Delta_2(C_2)$-definable. If $n = 2$, we are done, and otherwise, since $(I, (A_I)^{(n-3)}) \models \mathrm{B}\Sigma^0_3$ and $(I, (A_I)^{(n-2)} \oplus C_2) \models \mathrm{B}\Sigma^0_2$, we can use Theorem 1.1 again to get C_3 such that $(I, (A_I)^{(n-3)} \oplus C_3) \models \mathrm{B}\Sigma^0_3$ and C_1 is $\Delta_2(\Delta_2(C_3))$-, thus $\Delta_3(C_3)$-definable. Continuing in this way, we eventually get C_n such that $(I, A_I \oplus C_n) \models \mathrm{B}\Sigma^0_n$ and C_1 is $\Delta_n(C_n)$-definable. But then $(I, \Delta^0_1\text{-Def}(A_I \oplus C_n))$ is a model of $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n + \exists X \alpha(X)$. □

Corollary 4.2. *For any $n \geqslant 1$, the Π^1_1-consequences of* $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ *are contained in those of*

$$\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \{\mathrm{I}\Sigma^0_m \to \mathrm{B}\Sigma^0_{m+1} : m \in \omega\}. \tag{6}$$

The theory in (6) is one of two natural relativizations of the theory IB from [24]. In [26], this is referred to as the "weak" relativization.

Proof. Let $(M, \mathcal{X})$ be a model of the theory in (6) and let $A \in \mathcal{X}$ satisfy $\alpha(A)$ with α arithmetical. If $m \in \omega$ is the smallest such that $(M, \mathcal{X}) \not\models \mathrm{I}\Sigma^0_m$, then $m \geqslant n$ and $(M, \mathcal{X}) \models \mathrm{B}\Sigma^0_m$, so $(M, \Delta^0_{m-n+1}\text{-Def}(M, \mathcal{X})) \models \mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n + \exists X \alpha(X)$. On the other hand, if $(M, \mathcal{X})$ satisfies induction for all arithmetical formulas, then the closure of $\mathcal{X}$ under arithmetical definability witnesses that $\exists X \alpha(X)$ is consistent with ACA_0 and thus, by Proposition 4.1, with $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$. □

In order to generalize further results of Section 3, in particular Lemma 3.5 and Theorem 3.6, to arbitrary n, we need a suitable variant of Weak König's Lemma.

Definition 4.3. Let Δ^0_n-WKL be the statement: "for every Δ^0_n-set $\widetilde{T}$ that is an infinite 0–1 tree, there exists a Δ^0_n-set $\widetilde{W}$ that is an infinite path in $\widetilde{T}$".

Note that Δ^0_1-WKL is equivalent to WKL provably in RCA^*_0. On the other hand, Belanger [3] showed that Δ^0_2-WKL is equivalent to the cohesive set principle COH over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$. The appearance of $\mathrm{B}\Sigma^0_2$ here is not incidental. COH, being Π^1_1-conservative over RCA_0 [4], does not prove $\mathrm{B}\Sigma^0_2$, but an argument in the spirit of [13, Proposition 5] shows that Δ^0_2-WKL does.

Proposition 4.4. *For every $n \geqslant 1$, Δ^0_n-WKL implies $\mathrm{B}\Sigma^0_n$ over RCA^*_0. In fact, $\mathrm{B}\Sigma^0_n$ is already implied by the statement "for every Δ^0_n-set that is an infinite 0–1 tree and every number k, there is a node at level k in the tree with infinitely many nodes above it".*

Proof. For fixed n, we argue that the statement above implies $\mathrm{B}\Sigma^0_m$ by external induction on $m \leqslant n$. The thesis obviously holds for $m = 1$. Now assume that it holds for $m < n$ and that $\mathrm{B}\Sigma^0_{m+1}$ fails, and let $\psi(x, y)$ be a Π^0_m formula such that for some k, it holds that $\forall \sigma \in \{0,1\}^k\, \exists y\, \psi(\sigma, y)$ but the witnesses y cannot be bounded in a way independent of σ.

Consider the definable set of binary strings $\widetilde{T}$ consisting of all strings with length $\leqslant k$ and all $\sigma^\frown\tau$ where $|\sigma| = k$ and there is no $y \leqslant |\tau|$ with $\psi(\sigma, y)$. Then $\widetilde{T}$ is a Δ^0_{m+1}-set (in fact, a Σ^0_m-set) by $\mathrm{B}\Sigma^0_m$. Moreover, it is an infinite 0–1 tree, but for every σ of length k there are only finitely many vertices in $\widetilde{T}$ above σ. $\square$

Lemma 4.5. *For every $n \geqslant 1$, Δ^0_n-WKL is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n$. Moreover, any countable topped model of $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n$ can be ω-extended to a model of $\mathrm{RCA}^*_0 + \Delta^0_n$-WKL.*

Proof. For $n = 1$, this is [39, Corollary 4.7], and for $n = 2$, it follows from results of [6] and [3].

We prove the general case by induction on n. Like in the proof of Proposition 4.1, we use a jump inversion argument based on Theorem 1.1. Assume that the statement holds for n. To prove it for $n + 1$, it is enough to show that given any countable model $(M, A) \models \mathrm{B}\Sigma^0_{n+1}$ and a $\Delta_{n+1}(A)$-definable infinite 0–1 tree $\widetilde{T}$, we can find $B \subseteq M$ such that $(M, A \oplus B) \models \mathrm{B}\Sigma^0_{n+1}$ and there is a $\Delta_{n+1}(B)$-definable path in $\widetilde{T}$.

Note that $(M, A') \models \mathrm{B}\Sigma^0_n$ and $\widetilde{T}$ is $\Delta_n(A')$-definable. So, by our inductive assumption, there exists $\widetilde{P} \subseteq M$ such that $(M, A' \oplus \widetilde{P}) \models \mathrm{B}\Sigma^0_n$ and there is a $\Delta_n(\widetilde{P})$-definable path $\widetilde{G}$ in $\widetilde{T}$. By Theorem 1.1, there is some $B \subseteq M$ such that $(M, A \oplus B) \models \mathrm{B}\Sigma^0_{n+1}$ and $\widetilde{P}$ is $\Delta_2(B)$-definable. Thus, $\widetilde{G}$ is $\Delta_n(\Delta_2(B))$-definable, and hence $\Delta_{n+1}(B)$-definable because $\Delta_2(B)$-collection holds. $\square$

It follows immediately from Theorem 2.1 that any two countable models of Δ^0_n-WKL that ω-extend the same model of $\mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ are to some degree similar.

Corollary 4.6. *Let $n \geqslant 1$, and let $(M, \mathcal{X})$ and $(M, \mathcal{W})$ be countable models of $\mathrm{RCA}^*_0 + \Delta^0_n$-WKL such that $(M, \mathcal{X} \cap \mathcal{W}) \models \neg\mathrm{I}\Sigma^0_n$. Let $\bar{c}$ be a tuple of elements of M and $\overline{C}$ be a tuple of subsets of M that are Δ^0_n-definable in both $\mathcal{X}$ and $\mathcal{W}$. Then there exists an isomorphism h between $(M, \Delta^0_n\text{-}\mathrm{Def}(M, \mathcal{X}))$ and $(M, \Delta^0_n\text{-}\mathrm{Def}(M, \mathcal{W}))$ such that $h(\bar{c}) = \bar{c}$ and $h(\overline{C}) = \overline{C}$.*

Proof. Apply Theorem 2.1 to $(M, \Delta^0_n\text{-}\mathrm{Def}(M, \mathcal{X}))$ and $(M, \Delta^0_n\text{-}\mathrm{Def}(M, \mathcal{W}))$. $\square$

We introduce an auxiliary piece of notation.

Definition 4.7. Given $n \geqslant 0$ and a set A, we write $X \ll_A^n Y$ for the statement "for every $\Delta_n(X \oplus A)$-set $\widetilde{T}$ that is an infinite 0–1 tree, there exists a $\Delta_n(Y \oplus A)$-set $\widetilde{W}$ that is an infinite path in $\widetilde{T}$".

The choice of the $\ll$ symbol is inspired by the computability-theoretic notation $X \ll Y$, which means that Y has PA-degree relative to X, that is, every X-computable infinite 0–1 tree has a Y-computable path.

Lemma 4.8. *For each $n \geqslant 1$,* RCA_0^* *proves that:*

(a) for any sets X, Y, A, if $X \ll_A^n Y$, then $\mathrm{B}\Sigma_n(X \oplus A)$ holds,

(b) for any sets X, Y, A: $X \ll_A^n Y$ holds if and only if there exists a $\Delta_n(Y \oplus A)$-set that codes an ω-model of WKL_0^* *containing $(X \oplus A)^{(n-1)}$,*

(c) Δ_n^0-WKL is equivalent to $\forall X\, \forall Z\, \exists Y\, X \ll_Z^n Y$.

Proof. To prove (a), note that if $X \ll_A^n Y$, then in particular for every number k and every $\Delta_n(X \oplus A)$-set that is an infinite 0–1 tree, there is a node at level k with infinitely many nodes above it. By the argument from the proof of Proposition 4.4, this implies $\mathrm{B}\Sigma_n(X \oplus A)$.

We turn to (b). First assume that there is a $\Delta_n(Y \oplus A)$-set $\widetilde{W}$ coding an ω-model $\widetilde{\mathcal{W}}$ of WKL_0^* such that $(X \oplus A)^{(n-1)} \in \widetilde{\mathcal{W}}$. Clearly, every element of $\widetilde{\mathcal{W}}$ is a $\Delta_n(Y \oplus A)$-set. Moreover, every $\Delta_n(X \oplus A)$-set belongs to $\widetilde{\mathcal{W}}$, because $\widetilde{\mathcal{W}}$ contains $(X \oplus A)^{(n-1)}$and is closed under Δ_1^0-comprehension. If the $\Delta_n(X \oplus A)$-set happens to be an infinite 0–1 tree, it will have an infinite path belonging to $\widetilde{\mathcal{W}}$. So, $X \ll_A^n Y$ holds. In the other direction, if $X \ll_A^n Y$ holds, then by (a) the $\Delta_n(X \oplus A)$-sets form a model of RCA_0^*, so by Lemma 3.2 there is a single infinite $\Delta_n(X \oplus A)$-definable 0–1 tree $\widetilde{T}$ such that any $\widetilde{W}$ that is a path in $\widetilde{T}$ codes an ω-model of WKL_0^* containing $(X \oplus A)^{(n-1)}$. Since $X \ll_A^n Y$, some such $\widetilde{W}$ is a $\Delta_n(Y \oplus A)$-set.

In the proof of (c), the right-to-left direction is immediate. In the other direction, assuming Δ_n^0-WKL we get $\mathrm{B}\Sigma_n^0$ by Proposition 4.4. So, by Lemma 3.2 again, given sets X and Z there is a single infinite $\Delta_n(X \oplus Z)$-definable 0–1 tree $\widetilde{T}$ such that any path in $\widetilde{T}$ codes an ω-model of WKL_0^* containing $(X \oplus Z)^{(n-1)}$. By Δ_n^0-WKL, there is some Y for which there is a $\Delta_n(Y)$-definable path in $\widetilde{T}$. Then $X \ll_Z^n Y$ holds for any such Y. □

Definition 4.9 ($\ll_A^n$-basis theorem)**.** Let $n \geqslant 1$ and let ψ be a Π_2^1 sentence of the form $\forall X\, \exists Y\, \alpha(X, Y)$ where α is arithmetical. For a given set A, the $\ll_A^n$*-basis theorem for ψ* is the following statement:

> for any sets Z and X, if $X \ll_A^n Z$ then there exists a $\Delta_n(Z \oplus A)$-set $\widetilde{Y}$ such that $\alpha(X, \widetilde{Y})$ and $X \oplus \widetilde{Y} \ll_A^n Z$.

The following result can be viewed as a generalization of Lemma 3.5.

Theorem 4.10. *Let $n \geqslant 1$ and let ψ be a Π_2^1 sentence of the form $\forall X\, \exists Y\, \alpha(X, Y)$ where α is arithmetical. Then ψ is Π_1^1-conservative over* $\mathrm{RCA}_0^* + \mathrm{B}\Sigma_n^0 + \neg\mathrm{I}\Sigma_n^0$ *if and only if* $\mathrm{RCA}_0^* + \mathrm{B}\Sigma_n^0$ *proves the following Π_1^1 sentence γ_ψ^n:*

> *"for every set A, if* $\mathrm{I}\Sigma_n(A)$ *does not hold, then the $\ll_A^n$-basis theorem for ψ holds".*

We will write γ_ψ instead of γ^n_ψ whenever n is clear from the context, including in the proof of Theorem 4.10 itself.

Proof. The left-to-right direction can be proved by an argument that is similar to the one in the proof of Lemma 3.5, but a bit more complicated. We argue in a slightly different way in order to obtain some additional information (Corollary 4.11).

Let ψ be the Π^1_2 sentence $\forall X\, \exists Y\, \alpha(X,Y)$. We argue that the Π^1_1 sentence γ_ψ is provable in $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \psi$. Let $(M,\mathcal{X})$ be a countable model of that theory. Let $X,Z,A \in \mathcal{X}$ be such that $\neg\mathrm{I}\Sigma_n(A)$ holds and $X \ll^n_A Z$. Since ψ is true in $(M,\mathcal{X})$, there is a set $Y \in \mathcal{X}$ such that $(M,\mathcal{X}) \models \alpha(X,Y)$. By Lemma 4.5, we can ω-extend $(M, X \oplus Y \oplus Z \oplus A)$ to a structure $(M,\mathcal{Y})$ satisfying Δ^0_n-WKL. Of course, $(X \oplus Y \oplus A)^{(n-1)}$ is a Δ^0_n-set in $(M,\mathcal{Y})$.

The statement $X \ll^n_A Z$ is arithmetical in X,Z,A, so since it was true in $(M,\mathcal{X})$, it also holds in all the other second-order universes considered. So, Lemma 4.8(b) implies that there is a $\Delta_n(Z\oplus A)$-set $\widetilde{W}$ coding a model $(M,\widetilde{\mathcal{W}})$ of WKL^*_0 with $(X\oplus A)^{(n-1)} \in \widetilde{\mathcal{W}}$. By Δ^0_n-WKL (and Proposition 4.4), we know that $(M, \Delta^0_n\text{-Def}(M,\mathcal{Y}))$ is a model of WKL^*_0, and by $\neg\mathrm{I}\Sigma_n(A)$, we have $(M, \Delta^0_n\text{-Def}(M,\mathcal{Y})) \models \neg\mathrm{I}\Sigma^0_1$ and $(M,\widetilde{\mathcal{W}}) \models \neg\mathrm{I}\Sigma^0_1$.

Thus, by Corollary 2.2, the statement "there exists Y such that $\alpha(X,Y)$ and $(X\oplus Y\oplus A)^{(n-1)}$ exists as a set", which is true in $(M, \Delta^0_n\text{-Def}(M,\mathcal{Y}))$, must also be true in $(M,\widetilde{\mathcal{W}})$. Let $\widetilde{Y} \in \widetilde{\mathcal{W}}$ be a witness to the $\exists Y$ quantifier in that statement. Then in $(M, X\oplus Z\oplus A)$, and hence also in $(M,\mathcal{X})$, it is the case that $\widetilde{Y}$ is a $\Delta_n(Z\oplus A)$-set and $\alpha(X,\widetilde{Y})$ holds. Moreover, $X\oplus\widetilde{Y} \ll^n_A Z$, because each $\Delta_n(X\oplus\widetilde{Y}\oplus A)$-set belongs to $\widetilde{\mathcal{W}}$ which is an ω-model of WKL^*_0 coded by a $\Delta_n(Z\oplus A)$-set (cf. the proof of the right-to-left direction of Lemma 4.8(b), whose statement might not apply directly because $\widetilde{Y}$ might not be a set in the sense of $(M,\mathcal{X})$).

Since X,Z,A were arbitrary such that $\neg\mathrm{I}\Sigma_n(A)$ holds and $X \ll^n_A Z$, this completes the argument that $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \psi$ proves γ_ψ. By Π^1_1-conservativity, also $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ proves γ_ψ. Of course, $\mathrm{RCA}^*_0 + \mathrm{I}\Sigma^0_n$ proves γ_ψ as well by the definition of γ_ψ.

In the right-to-left direction, assume that $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n$ proves γ_ψ. By a standard ω-chain argument, to prove Π^1_1-conservativity of ψ over $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ it is enough to show that for any countable $(M,A) \models \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma_n(A)$ and any $\Delta_1(A)$-set X, there exists $\widetilde{Y} \subseteq M$ such that $(M, \widetilde{Y}\oplus A) \models \mathrm{B}\Sigma^0_n$ and $\alpha(X,\widetilde{Y})$ holds. By Lemma 4.5, we can extend (M,A) to a model $(M,\mathcal{X}) \models \mathrm{RCA}^*_0 + \Delta^0_n$-WKL. By Lemma 4.8(c), we can take some $Z \in \mathcal{X}$ such that $X \ll^n_A Z$. It follows from our assumption that $(M,\mathcal{X}) \models \gamma_\psi$, so there is some $\widetilde{Y} \subseteq M$ such that $\alpha(X,\widetilde{Y})$ holds and $X\oplus\widetilde{Y} \ll^n_A Z$. Since $Z\oplus A \in \mathcal{X}$, we have $(M,(Z\oplus A)^{(n-1)}) \models \mathrm{B}\Sigma^0_1$ and thus $(M,\widetilde{Y}\oplus A) \models \mathrm{B}\Sigma^0_n$, which is what we wanted to prove. □

We record that the proof of the left-to-right direction of the theorem actually shows the following.

Corollary 4.11. *Let $n \geqslant 1$ and let ψ be a Π^1_2 sentence. Then $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \psi$ proves γ^n_ψ, where γ^n_ψ is the Π^1_1 sentence from Theorem 4.10.*

A discussion of what Theorem 4.10 and Corollary 4.11 say about proving conservativity over collection principles, mostly in the context of Ramsey's theorem for pairs and two colours, can be found in Section 5.

The following Theorem can be viewed as a generalization of Theorem 3.6. Whereas Theorem 3.6 says that a Π^1_2 sentence ψ is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ exactly if it is provable from WKL, the result below replaces WKL with Δ^0_n-WKL and replaces provability of ψ with the provability of a more complicated Π^1_2 sentence guaranteeing that the second-order universe can be extended by solutions to instances of ψ. The sentence says that well-behaved but possibly non-set solutions to ψ exist, and that they can be found uniformly for instances of bounded complexity.

Theorem 4.12. *Let $n \geqslant 1$ and let ψ be a Π^1_2 sentence of the form $\forall X\, \exists Y\, \alpha(X, Y)$ where α is arithmetical. Then ψ is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ if and only if $\mathrm{RCA}^*_0 + \Delta^0_n$-WKL $+ \neg\mathrm{I}\Sigma^0_n$ proves the statement:*

$$\forall X_0\, \exists Y_0\, \forall X \leqslant_{\mathrm{T}} X_0\, \exists\, \Delta_n(Y_0)\text{-set}\, \widetilde{Y}\, \big(\widetilde{Y} \oplus X_0 \ll^n_\emptyset Y_0 \wedge \alpha(X, \widetilde{Y})\big). \tag{7}$$

Proof. Let ψ be Π^1_2 of the form $\forall X\, \exists Y\, \alpha(X, Y)$.

For the left-to-right direction, assume that ψ is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$, so $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n$ proves the sentence γ_ψ from Theorem 4.10. We argue within $\mathrm{RCA}^*_0 + \Delta^0_n$-WKL $+ \neg\mathrm{I}\Sigma^0_n$ and fix a set A satisfying $\neg\mathrm{I}\Sigma_n(A)$. Consider any set X_0. By Lemma 4.8(c), there is Y_0 such that $X_0 \ll^n_{X_0 \oplus A} Y_0$. Naturally, this implies $X \ll^n_{X_0 \oplus A} Y_0$ for every $X \leqslant_{\mathrm{T}} X_0$ as well. We have $\neg\mathrm{I}\Sigma_n(X_0 \oplus A)$, so we know by γ_ψ that the $\ll^n_{X_0 \oplus A}$-basis theorem for ψ holds. Therefore, for every $X \leqslant_{\mathrm{T}} X_0$ there exists a $\Delta_n(X_0 \oplus Y_0 \oplus A)$-set $\widetilde{Y}$ such that $\alpha(X, \widetilde{Y})$ and $\widetilde{Y} \oplus X \ll^n_{X_0 \oplus A} Y_0$; the latter implies $\widetilde{Y} \oplus X_0 \ll^n_\emptyset X_0 \oplus Y_0 \oplus A$. This proves (7) with the outermost $\exists$ quantifier witnessed by $X_0 \oplus Y_0 \oplus A$.

The proof of the right-to-left direction is just like the one in Theorem 4.10. □

The first of the two corollaries below follows directly from either Theorem 4.10 or Theorem 4.12. The second follows from their proofs in the right-to-left direction.

Corollary 4.13. *For each $n \geqslant 1$, the set of Π^1_2 sentences which are Π^1_1-conservative over $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ is c.e. Thus, it is computably axiomatizable.*

Except for $n = 1$, we do not know whether the set in question is finitely axiomatizable.

Corollary 4.14. *Let $n \geqslant 1$ and let ψ be a Π^1_2 sentence which is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$. Any countable topped model of $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ can be ω-extended to a model of $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n + \psi$.*

Remark. An analogue of Corollary 4.14 for $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n + \neg\mathrm{B}\Sigma^0_{n+1}$ fails. One way of showing this is as follows. Consider the theory in the language of first-order arithmetic and an additional predicate Tr with axioms consisting of $\mathrm{I}\Sigma_n$ and the statement that Tr is a truth class, i.e. satisfies Tarski's inductive conditions for a definition of truth for the arithmetical language. It is known that this theory is conservative over $\mathrm{I}\Sigma_n$ (this was proved for PA instead of $\mathrm{I}\Sigma_n$ in [33] and generalized to a wider class of theories in [36]; see also [9]). Therefore, every countable recursively saturated model of $\mathrm{I}\Sigma_n$ admits a truth class, and hence, by the results of [42] (see Theorem 6.3 in the present paper), the Σ^1_1 sentence "there is a Δ^0_{n+1}-set which is a truth class for the language of first-order arithmetic" is Π^1_1-conservative over $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n + \neg\mathrm{B}\Sigma^0_{n+1}$. On the other hand, it is also known that a nonstandard model of $\mathrm{I}\Sigma_n$ admitting a truth class has to be recursively saturated (this was originally proved for PA instead of $\mathrm{I}\Sigma_n$ in [35]; the new proof in [32], as noted in Remark 1 of that paper, works already over $\mathrm{I}\Delta_0 + \exp$ and thus over $\mathrm{I}\Sigma_n$ for each $n \geqslant 1$). As a consequence, a nonstandard model of $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n$ will not ω-extend to a model of the above Σ^1_1 sentence unless it is recursively saturated.

5 Arithmetical consequences of RT^2_2

In this section, we study the implications of our results for the general question of what methods can be used to prove Π^1_1-conservativity of Π^1_2 sentences over collection, and for the concrete problem whether Ramsey's Theorem for pairs and two colours (RT^2_2) is Π^1_1-conservative over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$. This is a major open problem in reverse mathematics, originally posed in [4].

Our main technical observation in this area is an upper bound on the quantifier complexity of the first-order part of the Π^1_1 sentences one has to consider to settle the problem about RT^2_2 (Corollary 5.2). We give two proofs of the bound, both of which work for a wider class of statements than just RT^2_2. The first proof is based on Theorem 4.10 and Corollary 4.11. The second actually avoids the use of Theorem 2.1 and its corollaries altogether, but relies heavily on the particular syntactic form of RT^2_2.

5.1 Implications of the isomorphism theorem

In [4], there are two separate approaches used to prove that every computable colouring $f\colon [\omega]^2 \to 2$ has an infinite homogeneous set H that is low_2, which means that $H'' \equiv_{\mathrm{T}} 0''$. The "double jump control" approach is to build H in such a way as to control H'' directly using $0''$. The "single jump control" approach is to show that if W is any set that has PA-degree relative to $0'$ – that is, to recall, if each $0'$-computable infinite 0–1 tree has a W-computable path – then there is H homogeneous for f such that W computes H', and in fact still has PA-degree relative to H'. This is essentially the unrelativized version of the $\ll^2_{\emptyset}$-basis theorem for RT^2_2, i.e. the special case of the $\ll^2_{\emptyset}$-basis theorem for $X = \emptyset$, with Z such that $Z' \equiv_{\mathrm{T}} W$. Since by the low basis theorem relativized to $0'$ there is a set W of PA-degree relative to $0'$ such that $W' \leqslant_{\mathrm{T}} 0''$, single jump control again gives H that is low_2.

A relativized version of the double jump control approach was the one used to prove Π^1_1-conservativity of RT^2_2 over $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$. When $\mathrm{I}\Sigma^0_2$ fails, single jump control (or the $\ll^2_{\emptyset}$-basis theorem) seems more likely to be applicable, and indeed it was applied in [6] to prove that weakenings of RT^2_2 known as CAC and ADS are Π^1_1-conservative over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$. *A priori*, however, it is conceivable that a proof of Π^1_1-conservativity of RT^2_2 over $\mathrm{B}\Sigma^0_2$ could use neither of the two approaches.

What Theorem 4.10 shows is that if a Π^1_2 sentence $\psi := \forall X \exists Y \alpha$ is Π^1_1-conservative over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$, then in principle this *has to* be provable by means of the $\ll^n_A$-basis theorem for ψ, where A is any witness to $\neg\mathrm{I}\Sigma^0_n$. In the specific case of RT^2_2, the relevant n equals 2, and conservativity over $\mathrm{I}\Sigma^0_2$ is already known from [4]. Thus, if RT^2_2 is in fact Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$, then in the currently unknown cases it will always be possible to apply the single jump control argument (relative to a set A witnessing $\neg\mathrm{I}\Sigma^0_2$).

The results of Section 4 also imply that there is always a bound on the complexity of Π^1_1 sentences we need to study in order to understand if a given Π^1_2 sentence is Π^1_1-conservative over $\mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$. For a Π^1_2 sentence ψ, it follows from Theorem 4.10 and Corollary 4.11 that ψ is Π^1_1-conservative over $\mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ if and only if $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n$ proves the sentence γ^n_ψ. If ψ is $\forall\exists\Pi^0_k$, then γ^n_ψ can be written as a $\forall\Pi^0_\ell$ statement for $\ell = \max(n+3, k+2)$; and, by Corollary 4.11, it is provable in $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n$. So, we get:

Corollary 5.1. *Let $n \geqslant 1$, and let ψ be a $\forall\exists\Pi^0_k$ sentence, where $k \geqslant 2$. Then ψ is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ if and only if it is $\forall\Pi^0_\ell$-conservative over $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$, where $\ell = \max(n+3, k+2)$.*

RT^2_2 is a $\forall\exists\Pi^0_2$ sentence, and we have the additional information that it is Π^1_1-conservative over $\mathrm{I}\Sigma^0_2$. This gives:

Corollary 5.2. $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ *is Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$ if and only if it is $\forall\Pi^0_5$-conservative over $\mathrm{B}\Sigma^0_2$.*

The discussion up to this point has only used the following properties of RT^2_2: it is a Π^1_2, and more precisely $\forall\exists\Pi^0_2$, sentence that is Π^1_1-conservative over $\mathrm{I}\Sigma^0_2$ and implies $\mathrm{B}\Sigma^0_2$ in the presence of RCA_0. More specific features of RT^2_2 are needed for the remark below.

Remark. In the special case of RT^2_2, the role of $\gamma^2_{\mathrm{RT}^2_2}$ in the preceding can be played by the $\ll^2_\emptyset$-basis theorem for RT^2_2. Indeed, we have the following:

(1) $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ proves the $\ll^2_\emptyset$-basis theorem for RT^2_2.

(2) $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ is Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$ if and only if $\mathrm{B}\Sigma^0_2$ proves the $\ll^2_\emptyset$-basis theorem for RT^2_2.

To show (1), let $(M, \mathcal{X})$ be a countable model of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$, and assume that $X, Z \in \mathcal{X}$ are such that $X \ll^2_\emptyset Z$ and X is a 2-coloring of $[M]^2$. (Note that $(M, \mathcal{X})$ is already a model of Δ^0_2-WKL.) By Lemma 4.8(b), there is a $\Delta_2(Z)$-set $\widetilde{W}$ coding a model $(M, \widetilde{\mathcal{W}})$ of WKL^*_0 with $X' \in \widetilde{\mathcal{W}}$. Applying Lemma 3.2, take $\widetilde{V} \in \mathcal{W}$ such that $X' \ll^1_\emptyset \widetilde{V}$. If $\mathrm{I}\Sigma_1(\widetilde{V})$ holds, then $\widetilde{V}$-primitive recursion is available in $(M, \mathcal{X})$. Thus one can directly formalize the single jump control argument in the form of the proofs of Theorems 6.44 and 6.57 and Corollary 6.58 of [18], which are carried out using solely "d-computable/primitive-recursive arguments" where d is a fixed degree such that $\emptyset' \ll d$. This gives a $\Delta_1(\widetilde{V})$-set $\widetilde{Y}$ such that $\widetilde{Y}$ is an infinite homogeneous set for X and $(X \oplus \widetilde{Y})' \leqslant_T \widetilde{V}$. Since $\widetilde{V} \ll^1_\emptyset Z'$, we get $X \oplus \widetilde{Y} \ll^2_\emptyset Z$. If $\mathrm{I}\Sigma_1(\widetilde{V})$ fails, then both $(M, \widetilde{\mathcal{W}})$ and $(M, \Delta^0_2\text{-Def}(M, \mathcal{X}))$ are models of $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ containing $\widetilde{V}$, so one may find $\widetilde{Y} \in \widetilde{\mathcal{W}}$ such that $\widetilde{Y}$ is an infinite homogeneous set for X and $(X \oplus \widetilde{Y})' \in \widetilde{\mathcal{W}}$

as in the proof of Theorem 4.10. This again implies $X \oplus \widetilde{Y} \ll^2_\emptyset Z$. Hence the $\ll^2_\emptyset$-basis theorem for RT^2_2 holds in $(M, \mathcal{X})$.

The left-to-right direction of (2) is a direct consequence of (1), and the right-to-left direction of (2) can be shown as in the proof of the right-to-left direction of Theorem 4.10.

5.2 Restricted Σ^1_1 formulas

Definition 5.3. A *restricted* Σ^1_1, or $\mathrm{r}\Sigma^1_1$, formula has the form $\exists Y \xi$, where ξ is Σ^0_3. A *restricted* Π^1_2, or $\mathrm{r}\Pi^1_2$, sentence has the form $\forall X\, (\eta(x) \to \exists Y \xi(X, Y))$, where $\exists Y \xi(X, Y)$ is $\mathrm{r}\Sigma^1_1$.

As mentioned above, RT^2_2 is a $\forall\exists\Pi^0_2$ and thus an $\mathrm{r}\Pi^1_2$ sentence.

In this subsection, we study the behaviour of $\mathrm{r}\Sigma^1_1$ formulas in $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$. We show that each $\mathrm{r}\Sigma^1_1$ formula is equivalent in $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ to an arithmetical formula with a relatively clear combinatorial meaning. Lifted to $n = 2$, this provides us with a $\forall\Pi^0_5$ consequence of RT^2_2 that leads to an alternative proof of Corollary 5.2, but differs from the sentence $\gamma^2_{\mathrm{RT}^2_2}$ used in the proof from Section 5.1 by having a meaningful restriction to computable instances.

The investigation of $\mathrm{r}\Sigma^1_1$ formulas also gives us an opportunity to discuss some additional interesting properties of $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$. We begin with a variation on a theme suggested by Theorem 2.1: the possibilities for changing a model of $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ by adding new sets to it are rather limited. Here, we show that it is not possible to add an unbounded set that is "sparser" than all those present in the ground model or to add a new bounded Σ^0_1-definable set. The statement about bounded sets is not used elsewhere in the paper but it is potentially of independent interest.

Lemma 5.4. *Let $(M, \mathcal{X}) \models \mathrm{RCA}^*_0$ and let $A \in \mathcal{X}$ be such that $(M, A) \models \neg\mathrm{I}\Sigma_1(A)$. Then:*

(a) For every unbounded set $S \in \mathcal{X}$ there exists an unbounded $\Delta_1(A)$-definable set B such that for every $b_1, b_2 \in B$ with $b_1 < b_2$ there exist $s_1, s_2 \in S$ with $b_1 \leqslant s_1 < s_2 \leqslant b_2$.

(b) Every bounded $\Sigma^0_1(M, \mathcal{X})$-definable set is $\Sigma_1(A)$-definable.

The result has a rather obvious generalization to higher n, in which $(M, \mathcal{X})$ satisfies $\mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma_n(A)$ and the statements (a) and (b) concern Δ^0_n- and Σ^0_n-definable sets, respectively.

Proof. Let C be an unbounded $\Delta_1(A)$-definable set such that $C = \{c_i : i \in I\}$ for some proper $\Sigma_1(A)$-definable cut $I \subsetneq_\mathrm{e} M$. Such a set C exists because $\neg\mathrm{I}\Sigma_1(A)$ holds.

We first prove (a). Given an unbounded $S \in \mathcal{X}$, we find the set B as a suitable subset of C. Define a sequence $\langle i_k : k \in K\rangle$ cofinal in I by

$$\begin{aligned} i_0 &= 0, \\ i_{k+1} &= \min\{i > i_k : |S \cap [c_{i_k}, c_i]| \geqslant 2\}. \end{aligned}$$

Here K consists of exactly those $k \in I$ for which i_k exists. K is clearly Σ^0_1-definable, and it is a cut because both S and C are unbounded sets. Let B be $\{c_{i_k} : k \in K\}$.

Clearly, B is unbounded and there are at least two elements of S between any two elements of B. It remains to show that B is $\Delta_1(A)$-definable. Notice that both $\{i_k : k \in K\}$ and $I \setminus \{i_k : k \in K\}$ are Σ^0_1-definable. So, by Theorem 1.2, there is some $d \in M$ such that $\mathrm{Ack}(d) \cap I = \{i_k : k \in K\}$. Then $B = \{c_i \in C : i \in \mathrm{Ack}(d)\}$, which shows that B is $\Delta_1(A)$-definable because C is.

For (b), we adopt some ideas from the proof of Theorem 2.2 in Chong–Yang [7]. Let R be a Σ^0_1-definable set in $(M, \mathcal{X})$ that is bounded above by $e \in M$. Suppose $R = \{x \in M : (M, \mathcal{X}) \models \exists z\, \theta(x, z)\}$, where θ is a Σ^0_0 formula, possibly with parameters from $(M, \mathcal{X})$. Use Theorem 1.2 to obtain $d \in M$ such that

$$\mathrm{Ack}(d) \cap (I \times [0, e]) = \{\langle i, x\rangle : (M, \mathcal{X}) \models \exists z \leqslant c_i\, \theta(x, z)\}.$$

Then $R = \{x \in M : x \leqslant e \wedge \exists i \in I\, \langle i, x\rangle \in \mathrm{Ack}(d)\}$. So R is $\Sigma_1(A)$-definable. $\square$

In general, an $\mathrm{r}\Sigma^1_1$ formula has the form $\exists Y\, \exists w\, \forall u\, \exists v\, \delta$ where δ is bounded. However, we can assume without loss of generality that the arithmetical part of the formula is actually $\forall u\, \exists v\, \delta$, because the initial existential quantifier $\exists w$ can be merged with the existential set quantifier $\exists Y$. Using standard tricks, we can also assume that all quantifiers in δ are bounded by v.

Definition 5.5. Let $\varphi(X)$ be an $\mathrm{r}\Sigma^1_1$ formula of the form $\exists Y\, \forall u\, \exists v\, \delta(X, Y, u, v)$, where all quantifiers in δ are bounded by v. We define $\alpha_\varphi(X, Z)$ to be the following arithmetical statement:

> There is an unbounded $\Delta_1(Z)$-definable set $W = \{w_i : i \in I\}$
> such that for every $i \in I$ there is a finite set $y \subseteq [0, w_i]$
> satisfying $\forall u \leqslant w_j\, \exists v \leqslant w_{j+1}\, \delta(X, y, u, v)$ for each $j < i$.

Roughly speaking, $\alpha_\varphi(X, Z)$ says that some Z-computable set provides a lower bound for the "rate of convergence" of sequence of finite approximations to a witness for the $\exists Y$ quantifier in $\varphi(X)$.

Theorem 5.6. *Let $\varphi(X)$ be $\mathrm{r}\Sigma^1_1$, and let $\alpha_\varphi(X, Z)$ be the arithmetical formula from Definition 5.5. Then:*

(a) $\mathrm{RCA}^*_0 \vdash \forall X\, \forall Z\, (\neg\mathrm{I}\Sigma_1(Z) \to (\varphi(X) \to \alpha_\varphi(X, Z)))$,

(b) $\mathrm{WKL}^*_0 \vdash \forall X\, \forall Z\, (\alpha_\varphi(X, Z) \to \varphi(X))$.

Proof. Let $\varphi(X)$ be $\exists Y\, \forall u\, \exists v\, \delta(X, Y, u, v)$ with quantifiers in δ bounded by v.

We first prove (b). Let X, Z be such that $\alpha_\varphi(X, Z)$ holds, and let the set $W = \{w_i : i \in I\}$ witness the existential quantifier in $\alpha_\varphi(X, Z)$. Let T be the tree consisting of finite 0–1 strings τ such that for every i satisfying $w_{i+1} < |\tau|$, we have $\forall u \leqslant w_i\, \exists v \leqslant w_{i+1}\, \delta(X, \{x : \tau(x) = 1\}, u, v)$. By the choice of W, the tree T is infinite, so by WKL there is an infinite path Y in T. Clearly, we have $\forall u \leqslant w_i\, \exists v \leqslant w_{i+1}\, \delta(X, Y, u, v)$ for each $i \in I$, which implies $\varphi(X)$.

Turning to (a), assume that we have $\varphi(X)$, and let Y witness the existential set quantifier in $\varphi(X)$. By $\mathrm{B}\Sigma^0_1$, for every number a there is some b such that $\forall u \leqslant a\, \exists v \leqslant b\, \delta(X, Y, u, v)$. This implies the existence of an unbounded set $\widehat{W} = \{\widehat{w}_j : j \in J\}$ such that for every $j \in J$ we have $\forall u \leqslant \widehat{w}_j\, \exists v \leqslant \widehat{w}_{j+1}\, \delta(X, Y, u, v)$.

Now let Z be such that $\neg\mathrm{I}\Sigma_1(Z)$ holds. By Lemma 5.4(a) with $S := \widehat{W}$, there exists an unbounded $\Delta_1(Z)$-definable set $W = \{w_i : i \in I\}$ such that for each $i \in I$ there is some $j \in J$ with $w_i \leqslant \widehat{w}_j < \widehat{w}_{j+1} \leqslant w_{i+1}$. As a consequence, we have $\forall u \leqslant \widehat{w}_i \, \exists v \leqslant \widehat{w}_{i+1} \, \delta(X,Y,u,v)$ for each $i \in I$. But this means in particular that W has the property required in $\alpha_\varphi(X,Z)$. $\square$

Corollary 5.7. *Let ψ be an $\mathrm{r}\Pi^1_2$ sentence of the form $\forall X\,(\eta(x) \to \varphi(X))$, where φ is $\mathrm{r}\Sigma^1_1$. Then ψ is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ if and only if RCA^*_0 proves $\forall X\,\forall Z\,(\neg\mathrm{I}\Sigma_1(Z) \wedge \eta(X) \to \alpha_\varphi(X,Z))$.*

We now consider what can be said about RT^2_2 using techniques based on Lemma 5.4. As in the proof of Corollary 5.2 in Section 5.1, the only specific features of RT^2_2 needed below are that it is an $\mathrm{r}\Pi^1_2$ sentence, implies $\mathrm{B}\Sigma^0_2$, and is Π^1_1-conservative over $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2$.

Let $\zeta(f,Z)$ express the following:

> If $f\colon [\mathbb{N}]^2 \to 2$, then there is an unbounded $\Delta_2(Z)$-set $\widetilde{W} = \{w_i : i \in I\}$ such that for every $i \in I$ there is a pair of finite strings $\langle \sigma, \tau \rangle$ satisfying:
>
> - $|\sigma| = |\tau| = w_i$,
> - $\{x : \sigma(x) = 1\}$ is homogeneous for f,
> - for each $j < i$, there are at least w_j elements $x \leqslant w_{j+1}$ such that $\sigma(x) = 1$,
> - for each $j < i$ and each $e \leqslant w_j$, if $\tau(e) = 1$, then there is a computation $s \leqslant w_{j+1}$ witnessing $e^{Z \oplus \sigma}\!\downarrow$, and if $\tau(e) = 0$, then $e^{Z \oplus \sigma}\!\uparrow$.

Loosely speaking, ζ says that, assuming f is a 2-colouring of pairs, there is a $\Delta_2(Z)$-definable infinite tree of finite approximations to an infinite homogeneous set for f and to the jump of the join of that set with Z. Note that ζ can be written as a Σ^0_4 formula, so the statement $\forall f\,\forall Z\,(\neg\mathrm{I}\Sigma_2(Z) \to \zeta(f,Z))$ is $\forall\Pi^0_5$.

Theorem 5.8. $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ *proves* $\forall f\,\forall Z\,(\neg\mathrm{I}\Sigma_2(Z) \to \zeta(f,Z))$. $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ *proves that statement if and only if* RT^2_2 *is* Π^1_1*-conservative over* $\mathrm{B}\Sigma^0_2$.

Proof. We first show that $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ proves $\forall f\,\forall Z\,(\neg\mathrm{I}\Sigma_2(Z) \to \zeta(f,Z))$. Let $(M,\mathcal{X}) \models \mathrm{RCA}_0 + \mathrm{RT}^2_2$. Let $f \in \mathcal{X}$ be a 2-colouring of pairs from M, and let $Z \in \mathcal{X}$ be such that $\mathrm{I}\Sigma_2(Z)$ fails. By RT^2_2, there is $H \in \mathcal{X}$ which is an infinite homogeneous set for f. Since $(M,\mathcal{X})$ satisfies $\mathrm{B}\Sigma^0_2$, there is an unbounded Δ^0_2-set $\widetilde{S} = \{s_j : j \in J\}$ in $(M,\mathcal{X})$ such that for each j, there are at least s_j elements of H below s_{j+1}, and each machine $e \leqslant s_j$ run with oracle $H \oplus Z$ either stops before s_{j+1} or does not stop at all.

By Lemma 5.4(a) applied to $(M, \Delta^0_2\text{-Def}(M,\mathcal{X}))$ with $S := \widetilde{S}$ and $A := Z'$, there exists an unbounded $\Delta_2(Z)$-definable set $\widetilde{W} = \{w_i : i \in I\}$ such that there are at least two elements of $\widetilde{S}$ between any two elements of $\widetilde{W}$. We claim that $\widetilde{W}$ witnesses that $\zeta(f,Z)$ holds. To see this, consider fixed $i \in I$. Let σ be the characteristic function of $H \cap [0,w_i]$, and let τ be the characteristic function of $(H \oplus Z)' \cap [0,w_i]$. Then the pair $\langle \sigma, \tau \rangle$ has the properties required by $\zeta(f,Z)$ for this i. This shows that $(M,\mathcal{X}) \models \forall f\,\forall Z\,(\neg\mathrm{I}\Sigma_2(Z) \to \zeta(f,Z))$.

Thus, we have proved that $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ implies $\forall f\,\forall Z\,(\neg\mathrm{I}\Sigma_2(Z) \to \zeta(f,Z))$. Now assume that $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ proves that statement as well. As in the proof of

Lemma 4.10, to prove the Π^1_1-conservativity of RT^2_2 over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2 + \neg\mathrm{I}\Sigma^0_2$, and thus over $\mathrm{RCA}_0+\mathrm{B}\Sigma^0_2$, it is enough to show that for any countable $(M, A) \models \mathrm{B}\Sigma^0_2 + \neg\mathrm{I}\Sigma^0_2$ and any $\Delta_1(A)$-definable 2-colouring of pairs f, there is $\widetilde{H} \subseteq M$ unbounded homogeneous for f such that $(M, A \oplus \widetilde{H})$ still satisfies $\mathrm{B}\Sigma^0_2$.

By [6], there is a model $(M, \mathcal{X}) \models \mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2 + \mathrm{COH}$ with $A \in \mathcal{X}$. By our assumption, $\zeta(f, A)$ holds in $(M, \mathcal{X})$. Since COH implies Δ^0_2-WKL over $\mathrm{RCA}_0+\mathrm{B}\Sigma^0_2$, there is a Δ^0_2-set in $(M, \mathcal{X})$ which is an infinite path in the infinite Δ_2-definable 0–1 tree provided by $\zeta(f, A)$. Thus, there is an unbounded Δ^0_2-set $\widetilde{H}$ in $(M, \mathcal{X})$ such that $\widetilde{H}$ is homogeneous for f and $(A \oplus \widetilde{H})'$ is a Δ^0_2-set in $(M, \mathcal{X})$. But this means that $(M, (A \oplus \widetilde{H})') \models \mathrm{B}\Sigma^0_1$, so $(M, A \oplus \widetilde{H}) \models \mathrm{B}\Sigma^0_2$. □

By Theorem 5.8, the statement $\forall f\, \forall Z\, (\neg\mathrm{I}\Sigma_2(Z) \to \zeta(f, Z))$ can be used to give an alternative proof of Corollary 5.2. A possible advantage of that statement over the sentence $\gamma_{\mathrm{RT}^2_2}$ is that the restriction of the latter to computable instances is trivially true in any model of $\mathrm{B}\Sigma_1 + \exp$, because in such a model there can never be computable sets A, Z, X such that $X \ll^n_A Z$ (cf. Lemma 3.8). On the other hand, the Π_5 sentence obtained by restricting $\zeta(f, Z)$ to computable colourings f and computable Z (that is, essentially, to situations where $\mathrm{I}\Sigma_2$ fails) seems more interesting, and it is quite unclear whether $\mathrm{B}\Sigma_2$ proves it.

6 Solution to a problem of Towsner

In [42], Towsner proved that for every $n \geqslant 1$, the set of Π^1_2 sentences ψ which are Π^1_1-conservative over $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n$ is Π_2-complete.

Towsner also asked whether the same holds with $\mathrm{I}\Sigma^0_n$ replaced by $\mathrm{B}\Sigma^0_n$. The question as stated is only really meaningful for $n \geqslant 2$, because $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_1$ is simply RCA_0. So, in order to generalize the question to $n = 1$ we take the liberty of changing the base theory to RCA^*_0.

Question 6.1 (essentially Towsner [42]). For fixed n, is the set

$$\{\psi \in \Pi^1_2 : \mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \psi \text{ is } \Pi^1_1\text{-conservative over } \mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n\}$$

Π_2-complete?

By Theorem 3.6 and Corollary 4.13, the answer to Question 6.1 is "almost negative", in that the set of Π^1_2 sentences ψ which are Π^1_1-conservative over $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ is c.e. for each n. For $n = 1$, we even know that this set is finitely axiomatizable.

Below we show that the original question, without the explicitly added $\neg\mathrm{I}\Sigma^0_n$, nevertheless has a positive answer.

Theorem 6.2. *For every $n \geqslant 1$, the set*

$$\{\psi \in \Pi^1_2 : \mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \psi \textit{ is } \Pi^1_1\textit{-conservative over } \mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n\}$$

is Π_2-complete.

To prove this, we recall an important result that Towsner uses as a lemma in his argument for the Π_2-completeness of Π^1_1-conservativity over $\mathrm{I}\Sigma^0_n$.

Theorem 6.3. *[42] If $(M, \mathcal{X})$ is a countable model of* $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n$ *and $\widetilde{S} \subseteq M$, then there is a family $\mathcal{Y} \supseteq \mathcal{X}$ of subsets of M such that $(M, \mathcal{Y}) \models \mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n$ and $\widetilde{S}$ is Δ^0_{n+1}-definable in $(M, \mathcal{Y})$.*

Definition 6.4. For each $n \geqslant 1$, the Σ^0_n *cardinality scheme*, $\mathrm{C}\Sigma^0_n$, asserts that no Σ^0_n formula defines a total injection from $\mathbb{N}$ to $\mathbb{N}$ with bounded range.

Theorem 6.5. *[26] For each $n, k \geqslant 1$, the theory* $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ *proves* $\mathrm{C}\Sigma^0_k$.

A proof of Theorem 6.5 is given in [26]. As mentioned in [26], a different proof from the one described there can be obtained by relativizing Kaye's proof of the result that any model of $\mathrm{B}\Sigma_1 + \exp + \neg\mathrm{I}\Sigma_1$ is elementarily equivalent to an $\aleph_\omega$-like structure [25, Theorem 2.4].

Note that Theorem 6.5 means that an analogue of Theorem 6.3 for $\mathrm{B}\Sigma^0_n$ fails. For example, if $(M, \mathcal{X})$ is a countable model of $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ then the graph of any bijection between M and the standard cut ω cannot be arithmetically definable in any $(M, \mathcal{Y}) \models \mathrm{B}\Sigma^0_n$ with $\mathcal{X} \subseteq \mathcal{Y}$.

In the proof of Theorem 6.2, we use a combination of Theorem 6.3 and Theorem 6.5.

Proof of Theorem 6.2. Fix $n \geqslant 1$. The set of those Π^1_2 sentences ψ that are Π^1_1-conservative over $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n$ is clearly Π_2, so we only need to prove completeness. Given a Π_2 sentence $\varphi := \forall x\, \exists y\, \delta(x, y)$, define the Σ^1_1 sentence ψ_φ as

$$\neg\mathrm{I}\Sigma^0_n \vee \exists Z\, \exists a\, \Big[(\text{there exists a } \Sigma_{n+1}(Z) \text{ function } \widetilde{f}\colon \mathbb{N} \hookrightarrow a) \wedge \forall x \leqslant a\, \exists y\, \delta(x, y)\Big].$$

We claim that ψ_φ is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n$ if and only if φ is true in the standard model of arithmetic ω.

Assume that φ is true in ω, and let $(M, \mathcal{X})$ be a countable nonstandard model of $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n$. If $(M, \mathcal{X}) \models \neg\mathrm{I}\Sigma^0_n$, then also $(M, \mathcal{X}) \models \psi_\varphi$. On the other hand, if $(M, \mathcal{X}) \models \mathrm{I}\Sigma^0_n$, by Theorem 6.3 we can ω-extend $(M, \mathcal{X})$ to $(M, \mathcal{Y})$ in which there is a Δ^0_{n+1}-definable (and thus Σ^0_{n+1}-definable) bijection $\widetilde{f}$ between M and ω. If $M \models \varphi$, let a be any nonstandard element of M. If $M \models \neg\varphi$, then, using $\mathrm{I}\Sigma_1$ in M, let a be the largest element of M such that $M \models \forall x \leqslant a\, \exists y\, \delta(x, y)$; in this case, a is necessarily nonstandard. In each case, we see that $\widetilde{f}$ is an injection from M into a_M, so $(M, \mathcal{Y}) \models \psi_\varphi$.

We have shown that if $\omega \models \varphi$, then every countable nonstandard model of $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n$ ω-extends to a model of $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \psi_\varphi$. This is enough to show that ψ_φ is Π^1_1-conservative over φ.

Now assume that φ is false in ω, and let $m \in \omega$ be such that $\omega \models \neg\exists y\, \delta(m, y)$. We argue that the Σ^1_1 formula $\neg\exists y\, \delta(m, y) \wedge \neg\mathrm{C}\Sigma^0_{n+1}$ is consistent with $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n$ (in fact, with $\mathrm{RCA}^*_0 + \mathrm{I}\Sigma^0_n$) but not with $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \psi_\varphi$.

Let M be a countable nonstandard model of $\mathrm{Th}(\omega)$. Then $(M, \Delta_1\text{-Def}(M)) \models \mathrm{RCA}^*_0 + \mathrm{I}\Sigma^0_n + \neg\exists y\, \delta(m, y)$. Use Theorem 6.3 to ω-extend $(M, \Delta_1\text{-Def}(M))$ to $(M, \mathcal{Y}) \models \mathrm{RCA}^*_0 + \mathrm{I}\Sigma^0_n + \neg\mathrm{C}\Sigma^0_{n+1}$. Of course, $\neg\exists y\, \delta(m, y)$ still holds in $(M, \mathcal{Y})$, because it is a purely arithmetical statement. This shows the consistency of $\neg\exists y\, \delta(m, y) \wedge \neg\mathrm{C}\Sigma^0_{n+1}$ with $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n$.

On the other hand, assume that a structure $(M, \mathcal{X})$ satisfies both $\mathrm{RCA}^*_0 + \mathrm{B}\Sigma^0_n + \psi_\varphi$ and $\neg\exists y\, \delta(m, y) \wedge \neg\mathrm{C}\Sigma^0_{n+1}$. By Theorem 6.5, $(M, \mathcal{X})$ must satisfy $\mathrm{I}\Sigma^0_n$, so it must also satisfy the second disjunct of ψ_φ. In particular, for any element a witnessing the existential number quantifier in that disjunct, we have $M \models \forall x \leqslant a\, \exists y\, \delta(x, y)$. On the other hand, any such a also has to be nonstandard, which gives a contradiction with $M \models \neg\exists y\, \delta(m, y)$. □

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